Social networks, commuting and (un)employment rates*

Provisional title

Job Market Paper

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Abstract

This paper proposes a partial equilibrium model of job search in a multicentric economy where job-seekers leave unemployment through their social network and the market. Workers and firms are heterogeneous in their geographical location and hence workers commute to their work place. Workers learn from open vacancies through their strong and weak ties or directly contacting firms. In this setup job-search activities involve deciding how far to search conditional on the worker's social network. Choosiness, defined as the job-search area, varies across individuals depending on the employment status of their strong tie. Job-seekers whose strong tie is employed are choosier than those whose strong tie is unemployed. However, when her strong tie is unemployed, employed weak ties become more valuable to the job search process. Employment (unemployment) rates rise (fall) with a larger job-search area. The effect of weak ties on employment rates is yet ambiguous.

Keywords: Strong and weak ties, commuting costs, employment rates.

JEL Classification: A14, J22, J64.

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1. Introduction

This article examines (un)employment rates in a multicentric economy, where job searching is determined by workers' geographical locations and social networks. Job seekers consider the distance to work and the possible channels employment, including social networks and the market. We follow Granovetter (1973) and model the social network by using strong (e.g. family) and weak (e.g. acquaintance) ties. We find that unemployed individuals with an employed strong tie are more *choosy*[1] than are unemployed individuals with an unemployed strong tie, meaning that they differ in their job search areas. While employment rates increase with larger job search areas, the effect of weak ties on employment rate is ambiguous. The mechanism behind relies on the negative effect of workers' weak ties on their choosiness.

Several studies argue that the distance to work is a valid criterion for workers to accept employment (see Stancanelli, Rupert, and Wasmer, 2009). Many papers also argue that workers not only contact firms but also approach friends and family as part of the strategy to leave unemployment. The main motivation of this work relates to Bayer, Ross, and Topa (2008) who find that residing in the same block raises the probability of sharing the work location by 33%, a finding that is consistent with local referral effects. Hellerstein, McInerney, and Neumark (2011) build on Bayer et al. (2008) and find similar results for ethnic minorities in the United States. In Schmutte (2014) workers who live in neighborhoods with high-quality networks (high-paying jobs) are more likely to find jobs with high wage premia than are workers who live in neighborhoods with low-quality networks.

We build a theoretical model in a multicentric setup[2] with exogenous labor demand. Workers and firms are uniformly distributed on the circumference of a circle of unit one. Workers can be employed or unemployed. When employed, they incur commuting costs; when unemployed, they search for a job. Individuals belong to a mutually exclusive 2-person dyad. Members of the same dyad have strong ties (family and close friends) and weak ties (acquaintances). Each dyad can be in the following states: (i) both workers are employed; (ii) one worker is employed and one worker is unemployed; and (iii) both

[1] This term has also been used in (Decreuse, 2008) for the productivity space.

[2] According to (Glaeser, 2007) workers and firms are geographically dispersed: "America changed from a nation of distinct cities separated by farmland, to a place where employment and population density is far more continuous."

workers are unemployed. All (unskilled) workers obtain the same wage and thus are not motivated to find new jobs. Nevertheless, employed workers are better informed about available employment opportunities within their firms, so they give that information to the first unemployed contact in their network, who can be a strong tie or weak tie. This information transmission protocol defines a Markov process. Because of the continuous time Markov process, members of a dyad cannot change status at the same time.

Literature review

This paper integrates previous literature on urban, labor and network economics. In urban economics, Gautier and Zenou (2010) use the notion of geographical space in a circle economy, to propose a model in which black and white workers reside on the circumference of a circle and differ only in their initial wealth. In their model, commuting by car is faster than using public transportation. Therefore, whites and blacks have different job acceptance thresholds. In our model, however, all workers are homogeneous, and we focus on job-search channels (i.e., the workers' social network and the market). Moreover, the worker's level of choosiness or area of search depends on the worker's social network. In labor economics, we are close to standard search-matching models, in which individuals choose reservations wages by comparing the values of employment and unemployment at the margin, also known as stopping rule. In the present model, individuals are spatially heterogeneous and the decision rule applies to the search area. Our article also relates to Zenou (2015) who studies employment rates when workers learn of vacancies through their social network and through the market at the same rate. We depart from this article by introducing individuals geographic location to better understand job-seekers' behavior.

Empirical evidence reveals that job-seekers use their social networks to find a job (e.g. Topa, 2001; Wahba and Zenou, 2005, among others). In Sweden, more than 60% of the vacancies were filled through informal channels (Hensvik and Nordström Skans, 2013). In Britain there is evidence that the number of employed friends an individual has influences his ro her chances of finding a job. Each additional employed friend increases the job-finding probability by as much as 13% or 3.3 percentage points (Cappellari and Tatsiramos, 2013). Kramarz and Nordström Skans (2014) find that, in Sweden, strong ties, (e.g. parents), are an important determinant for where young workers find their first job.

The notion of social networks in the labor market in the form of strong and weak ties includes studies by Giulietti, Wahba, and Zenou (2014) for the case of migration in China, Sato and Zenou (2014) and Patacchini and Zenou (2008) for the case of crime rates. To the best of our knowledge, no theoretical papers investigate social interactions in the labor market and embedded in a multicentric urban space.

The rest of the article is organized as follows. Section 2 presents the basic model framework. Section 3 provides a numerical analysis. Section 4 describes an extension of the basic model. Section 5 presents a discussion of the possible applications of the model. Section 6 concludes.

2. The model

2.1. The environment

Large cities tend to deconcentrate from a Central Business District (CBD) to several job centers. In fact, close to 50% of American Metropolitan Statistical Areas (MSAs) present polycentric employment structures (Arribas-Bela and Sanz-Gracia, 2014). Motivated by these characteristics we model an economy in continuous time and at steady state, as in Salop (1979). There is a continuum of two types of economic agents, workers and firms, which are uniformly distributed on the circumference C of a circle of length one.[3] Thus, workers and firms are *ex ante* heterogeneous in terms of their geographic location. An exogenous finite number of firms enters the market. All economic agents are risk neutral and infinitely-lived, and they discount the future at a common rate r. The total population is normalized to unity. Individuals are either employed and specialized in production or unemployed and specialized in job search.[4] Let i denote workers' location and by and j denote firms' locations. Thus, $x^{i,j}$ denotes the geographical distance from workers' residences to their respective work or job-search location. Consequently, commuting to work emerges as a necessity. Only employed workers incur commuting costs $\tau > 0$.[5] The maximum distance between worker and a firm locations is $x^{i,j} = 1/2$; hence, $0 < x^{i,j} \leqslant$

[3]Gautier and Zenou (2010) also study the geographic space along a circle C of unit length. Other studies in the circle economy consider the productivity space (Marimon and Zilibotti, 1999; Decreuse, 2008) and the skill space (Hamilton, Thisse, and Zenou, 2000; Helsey and Strange, 1990; Brueckner, Thisse, and Zenou, 2002).

[4]In this model, we do not consider on-the-job search.

[5]More precisely, τ denotes the pecuniary and time cost per unit of distance commuted to the work place.

1/2. We assume high relocation costs,[6] that prevent workers from moving closer to their work places. Another argument for high relocation costs is related to homes mobile, which are less mobile than jobs (Manning, 2003).

2.2. The social network

A simple way to model social networks is by considering pairs of individuals, namely dyads. In our model, each individual belongs to a dyad in which the two members share a strong relationship. Dyad members can be relatives or close friends and their relationship is defined as *strong ties*. Members of a dyad, (e.g. siblings) remain linked forever. Individuals do not spend all their time with their inner circle, however, so they also meet neighbors, colleagues, or other acquaintances who are henceforth defined as *weak ties*. Since individual employment status changes over time, we can observe dyads, denoted as d, in the three different states.

Table 1
Types of dyads

Dyad	Description
d_2	$\{employed, employed\}$
d_1	$\{employed, unemployed\}$, $\{unemployed, employed\}$
d_0	$\{unemployed, unemployed\}$

As shown in Table 1, the order of employment status within the dyad is irrelevant, (i.e., d_1-dyads).

All jobs and workers are identical (unskilled labor), hence all employed workers obtain the same exogenous wage. All employed workers receive information about open vacancies at an exogenous rate $\lambda > 0$, and they lose their job at the exogenous rate $\delta > 0$. At every period, when they learn of a position, they transmit this information to the individual they meet, regardless of that individual's tie strength. The only circumstance in which workers do not report the opening and instead keep the information to themselves is for higher wages or higher expected utility. We rule out such behavior by imposing homogeneous wages and by assuming that employed workers gather information from their employers. Since the opening is to work in the same firm and since this implies the same commuting distance (as explained below), workers lack incentive to leep this data to themselves.

[6]Other papers that assume high relocation costs are Zenou (2006, 2009b).

We assume that unemployed workers gather information about vacancies directly from employers at an exogenous rate $\psi > 0$.

We denote employment and unemployment at time t as $e(t)$ and $u(t)$ respectively, where $e(t)$, $u(t) \in [0, 1]$. The total population is

$$\underbrace{2d_0(t) + d_1(t)}_{u(t)} + \underbrace{d_1(t) + 2d_2(t)}_{e(t)} = 1.$$

Individuals randomly meet a weak or strong tie with exogenous probability ω and $1-\omega$, respectively, which can also be understood as the share of time spent with weak and strong ties. We assume that when a meeting takes place, it is reciprocal between two individuals. Hence the network is undirected (using graph theory terms, as in Jackson, 2008). This information transmission is in continuous time. At each period of time the transit between dyads depends on the employment status of the individuals whom they meet and the labor market turnover. Therefore, this is a Markov process, so dyads cannot transit from d_0 to d_2 or vice versa.

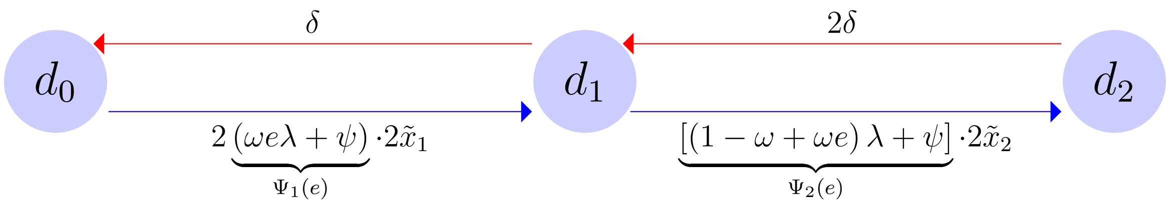

Fig. 1. Labor market transitions

Fig. 1 presents the flow of dyads between the different states, where each node represent the number of dyads in each state. Let us explain how dyads in d_0 transit to d_1 and d_2 states. Individuals that belong to a dyad in state $d_0 = \{unemployed, unemployed\}$, leave unemployment at a rate $2\Psi_1(e) \cdot 2\tilde{x}_1$, where $\Psi_1(e)$ denotes the rate at which each partner leaves unemployment and $2\tilde{x}_1$ denotes the job search area. In this setup, $\Psi_1(e)$ encompasses the rate at which a job-seeker meets an employed weak tie who has information about an open position ($\omega e \lambda$) plus the rate at which she directly contacts a firm with an open position (ψ). When one member of the d_0-dyad finds a job, this dyad moves to the state $d_1 = \{employed, unemployed\}$. Here, the unemployed members of the dyads find employment at a rate $\Psi_2(e) \cdot 2\tilde{x}_2$. $\Psi_2(e)$ includes the rate at which they meet their employed strong tie ($(1-\omega)\lambda$), plus the rate at which they meet an employed weak tie ($e\omega\lambda$) and the rate at which they find a job through the market (ψ). All workers face the same job destruction rate and leave d_2 and d_1 states at δ.

The transition probabilities of dyads across the three different states between t and $t+dt$ is written as:

$$\begin{cases} \dot{d}_2(t) = 2\tilde{x}_2\Psi_2(e(t))d_1(t) - 2\delta d_2(t) & (1) \\ \dot{d}_1(t) = 4\tilde{x}_1\Psi_1(e(t))d_0(t) - \delta d_1(t) - 2\tilde{x}_2\Psi_2(e(t))d_1(t) + 2\delta d_2(t) & (2) \\ \dot{d}_0(t) = \delta d_1(t) - 4\tilde{x}_1\Psi_1(e(t))d_0(t) & (3) \end{cases}$$

The evolution of the number of dyads in each state is given by the difference between the entry and exit flows. The variation of d_2-dyads in Eq. (1) is equal to the number of d_1-dyads that leave that state at a rate $2\tilde{x}_2\Psi_2(e(t))$ minus the number of d_2-dyads with a member who exits employment at a rate δ. The variation of d_1-dyads in Eq. (2) is equal to the number of d_0-dyads with a member who finds work at a rate $4\tilde{x}_1\Psi_1(e(t))$ minus the number of d_1-dyads with a member who loses a job at a rate δ, minus the number of d_1-dyads that transit to the d_2 state at a rate $2\tilde{x}_2\Psi_2(e(t))$ (because the unemployed member finds work), plus the number of d_2 dyads in which one member becomes unemployed. Last, the variation of d_0-dyads in Eq. (3) is equal to the number of d_1 dyads in which the employed member loses a job minus the number of d_0 dyads in which one of the members exit unemployment at a rate $2\tilde{x}_1\Psi_1(e(t))$.

2.3. Labor market equilibrium

In steady state, we set the net flows (1)-(3) equal to zero. This leads to the following equilibrium equations:

$$\tilde{d}_2 = 4\tilde{x}_1\tilde{x}_2 \cdot \frac{\Psi_1(\tilde{e})\Psi_2(\tilde{e})}{\delta^2} \cdot \tilde{d}_0 \quad (4)$$

$$\tilde{d}_1 = 4\tilde{x}_1 \cdot \frac{\Psi_1(\tilde{e})}{\delta} \cdot \tilde{d}_0 \quad (5)$$

$$\tilde{d}_0 = \frac{1}{2} - \tilde{d}_2 - \tilde{d}_1 \quad (6)$$

$$\tilde{u} = 1 - \tilde{e} \quad (7)$$

$$\tilde{e} = 2\tilde{d}_2 + \tilde{d}_1 \quad (8)$$

Assumption 1.

$$\psi > \frac{\omega\lambda}{2}.$$

where $0 < \omega < 1$.

Assumption 1 states that, conditional on the rate at which individuals meet a weak tie $\omega \in (0, 1)$, the rate of direct contact with vacancies in this model has to be greater than the rate of contact through weak ties divided by two. As explained below, this assumption is crucial for the characterization of the equilibrium. Moreover, empirical evidence suggests that more than 60% of positions are filled through contacts (Hensvik and Nordström Skans, 2013), meaning that $\psi < \lambda$, which supports our assumption.

Lemma 1. *At steady state, the labor market equilibrium is a tuple* $\{\tilde{d}_2, \tilde{d}_1, \tilde{d}_0, \tilde{u}, \tilde{e}\}$ *such that Eqs.* (4) - (8) *are satisfied and employment is given by the implicit function*

$$\mathcal{E}\left(\tilde{e}\right) = \alpha_3\, \tilde{e}^3 + \alpha_2\, \tilde{e}^2 + \alpha_1\, \tilde{e} - \alpha_0 = 0 \tag{9}$$

where:

$$\begin{aligned}
\alpha_3 &= 4\tilde{x}_1\tilde{x}_2\omega^2\lambda^2 > 0 \\
\alpha_2 &= 4\tilde{x}_1\omega\lambda\left\{\tilde{x}_2\left[(1-\omega)\,\lambda + \psi\right] + \delta + \tilde{x}_2\psi - \tilde{x}_2\omega\lambda\right\} > 0 \\
\alpha_1 &= \delta^2 + 4\tilde{x}_1\psi\left\{\tilde{x}_2\left[(1-\omega)\,\lambda + \psi\right] + \delta\right\} - 2\tilde{x}_1\omega\lambda\left\{2\tilde{x}_2\left[(1-\omega)\,\lambda + \psi\right] + \delta + 2\tilde{x}_2\psi\right\} > 0 \\
\alpha_0 &= 2\tilde{x}_1\psi\left\{2\tilde{x}_2\left[(1-\omega)\,\lambda + \psi\right] + \delta\right\} > 0
\end{aligned}$$

Proof. Substituting Eqs. (4) and (5) into (6), we obtain:

$$\tilde{d}_0 = \frac{1}{2} \cdot \frac{\delta^2}{\delta^2 + 4\tilde{x}_1\Psi_1(\tilde{e}) \cdot \left[\tilde{x}_2\Psi_2(\tilde{e}) + \delta\right]}. \tag{10}$$

By substituting Eqs. (4), (5) and (G.1) into Eq. (8), we obtain the implicit function (9). Moreover, by using Assumption 1, we show that α_3, α_2, α_1, $\alpha_0 > 0$. See Appendix A for computational details. □

Once we solve for employment,[7] conditional on $\tilde{x}_1$ and $\tilde{x}_2$, the characterization of the labor market equilibrium is recursive. First, we compute the number of d_0-dyads, value that determines the number of d_2-dyads and d_1-dyads, respectively. Then we can compute unemployment.

[7] Given that the population is normalized to unity, we use (un)employment and (un)employment rate interchangeably.

Table 2
Comparative statics of the dyads

	$\tilde{x}_1$	$\tilde{x}_2$	$\Psi_1(\tilde{e})$	$\Psi_2(\tilde{e})$	ω	e	λ	ψ
$\tilde{d}_2$	+	+	+	+	?	+	+	+
$\tilde{d}_1$	+	-	+	-	?	-	?	-
$\tilde{d}_0$	-	-	-	-	?	-	-	-

Table 2 presents an analysis of the comparative statics of the different dyads. As expected the number of $d_2 = \{employed, employed\}$ dyads increases with the job search area of d_0 and d_1 dyads, $\tilde{x}_1$ and $\tilde{x}_2$, respectively. Higher employment, a higher rate at which an employed worker hears about a position, or a higher rate of directly contacting an employer increases the number of dyads in state d_2. Conversely, these results are totally opposite for the number of $d_0 = \{unemployed, unemployed\}$ dyads. The number of dyads in state $d_1 = \{employed, unemployed\}$ increases when individuals in state d_0 increase their job search areas ($\tilde{x}_1$) and decreases when individuals in state d_1 increase their job search areas ($\tilde{x}_2$). A rise in employment and a higher firm contact rate (by the unemployed) reduces the number of individuals in the d_1 state. The same occurs with the number of individuals in the d_0 state. However, when the strong and weak ties who are employed hear about a vacancy, the effect is unclear. Two effects are in place. On the one hand, a rise in λ increases both global rates to leave unemployment ($\Psi_1(e)$ and $\Psi_2(e)$), but these rates have opposite effects on the number of dyads in the d_1 state. Moreover, changes in the social network parameter ω yield ambiguous effects on dyads in the three states, because $\frac{\partial \Psi_1(e)}{\partial \omega} > 0$ while $\frac{\partial \Psi_2(e)}{\partial \omega} = -1 + e\lambda \lesseqgtr 0$. See Appendix G.1 for computational details.

Proposition 1. *For $\tilde{x}_1, \tilde{x}_2 \in (0, 1/2]$ there exists a unique steady-state interior equilibrium employment rate $0 < \tilde{e}_1 < 1$.*

Proof. By Lemma 1, we know that α_3, α_2, α_1, $\alpha_0 > 0$, hence by the Descartes' sign rule, we conclude that the cubic function (9) has one real positive and two real negative roots or two imaginary roots. To check if the positive root is between zero and one, we find for $\mathcal{E}(1) = \delta + 2\tilde{x}_1\psi + 2\tilde{x}_1\omega\lambda > 0$ and $\mathcal{E}(0) = -\alpha_0 < 0$, meaning that the only positive root, denoted as $\tilde{e}_1$, is between zero and one. See Appendix B for computational details. □

At this point, it is crucial to find an expression for $\tilde{e}_1$, since it will be useful for

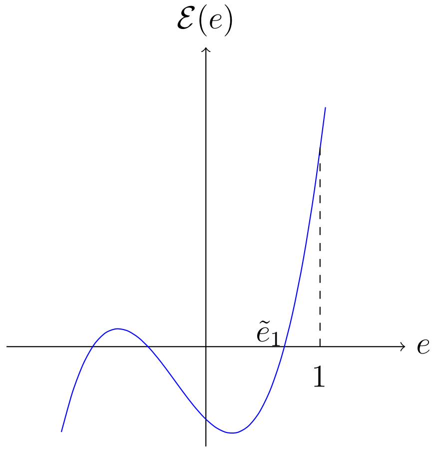

Fig. 2. The employment rate

the characterization of the equilibrium. Therefore, we use a trigonometric approach to explicitly write the positive root of (9).

Proposition 2. *Employment* $0 < \tilde{e}_1 < 1$ *is given by*

$$\tilde{e}_1 = 2\, g \cos\theta - \frac{a_2}{3} \tag{11}$$

where $\tilde{e}_1 \equiv \tilde{e}_1 (\tilde{x}_1, \tilde{x}_2)$*,* $g \equiv g (\tilde{x}_1, \tilde{x}_2)$*,* $\theta \equiv \theta (\tilde{x}_1, \tilde{x}_2)$ *and* $a_2 \equiv a_2 (\tilde{x}_1, \tilde{x}_2)$ *with* $\tilde{x}_1, \tilde{x}_2 \in (0, 1/2]$.

Proof. See Appendix C for the proof. Fig. 2 presents the implicit function of employment (9). □

When workers expand the distance that they are willing to commute to work, $\tilde{x}_1$ and $\tilde{x}_2$, they are more likely to leave unemployment and thus employment rises. The marginal impact of the time spent with employed weak ties, ω, increases employment.

2.4. Job search behavior: Myopic dyads

When the space dimension of economic agents is made explicit, job-seekers decide how far to search for employment. In this subsection, this decision is taken by myopic individuals. When myopic workers look far into the future, they expect the state of the opponent process to remain unchanged with high probability, so that the future looks like the present.

Let $W_{1,1}^{i,j}$ $\left(\text{resp. } W_{1,0}^{i,j}\right)$ be the steady state expected and discounted lifetime utility of an employed worker with residence in i and employment in j, whose partner is employed (resp. unemployed). Similarly, let $W_{0,1}^{i,j}$ $\left(\text{resp. } W_{0,0}^{i,j}\right)$ be the steady state expected and discounted lifetime utility of an unemployed worker with residence in i and searching in j, whose partner is employed (resp. unemployed). We write the steady state expected and discounted lifetime utility of an individual in the following compact form $W_{a,b}^{i,j}$ with $a, b \in \{1, 0\}$. In this setup, assuming a uniform distribution of workers over the circumference of a circle has simplifying consequences. We follow Lemma 1 of Marimon and Zilibotti (1999)[8] to show that because workers and firms are uniformly distributed there exist a symmetric equilibria, which in our case implies $W_{a,b}^{i,j} = W_{a,b}$ and $x^{i,j} = x$ $\forall i$ and $\forall j \in C$. In other words, given the uniform distribution of all economic agents over C, deviating from the equilibrium offers not benefit.

Myopic workers dismiss the expected loss (gain) of losing (finding) employment or the expected utility coming form their dyad members. The Bellman equations are written for the individual with the first subscript, i.e., W_{11} is the lifetime expected value of an employed worker whose strong ties is also employed.

$$rW_{11}(x) = w - \tau \cdot x \tag{12}$$

$$rW_{01} = b \tag{13}$$

$$rW_{10}(x) = w - \tau \cdot x \tag{14}$$

$$rW_{00} = b \tag{15}$$

The maximum commuting distance job-seekers are willing to accept is determined by the following conditions

$$\begin{aligned} W_{11}(x) &= W_{01} \quad \Leftrightarrow \quad \tilde{x}_2, \text{ when her strong tie is employed and} \\ W_{10}(x) &= W_{00} \quad \Leftrightarrow \quad \tilde{x}_1, \text{ when her strong tie is unemployed.} \end{aligned}$$

Hence, when individuals are myopic they search equally in spite of their strong tie employment status. That is

$$\tilde{x}_1 = \tilde{x}_2 = \frac{w - b}{\tau} \tag{16}$$

First, we observe that the job search areas depend only on exogenous parameters and by a simple numerical exercise, where $w = 1$, $b = 0.5$ and $\tau = 0.4$, the job search areas are larger than 1/2. This means that when individuals are myopic they search at the maximum possible distance from their residence; i other words, they are no longer choosy.

[8]This procedure has also been applied in Gautier and Zenou (2010)

Definition 1. *Myopic individuals. An Equilibrium is a given by the labor market equilibrium, Eq.* (11) *and the job search areas, given by* (16).

2.5. Job search behavior: Imperfectly forward looking dyads

This decision is taken by forward-looking individuals who anticipate the effect of their current decisions on their future payoffs.[9] However, these workers are myopic when it comes to anticipating the status of their current partner. In this subsection, we treat the partner's state as given. In Section 4, we present the case where individuals are forward-looking both for their own status and that of their strong tie.

Each Bellman equations is written for the individual with the first subscript. For example, $W_{11}(x)$ is the lifetime expected value of an employed worker whose distance to work x and whose strong ties is employed. W_{01} is the lifetime expected value of an unemployed worker whose strong tie is employed. Observe that W_{01} is not a function of distance, since job seekers do not need to commute in order to search for employment.

$$
\begin{aligned}
rW_{11}(x) &= w - \tau \cdot x - \delta \left[W_{11}(x) - W_{01} \right] && (17) \\
rW_{01} &= b + \Psi_2(e) \int_{-\tilde{x}_2}^{\tilde{x}_2} \left[W_{11}(x) - W_{01} \right] dx && (18) \\
rW_{10}(x) &= w - \tau \cdot x - \delta \left[W_{10}(x) - W_{00} \right] && (19) \\
rW_{00} &= b + \Psi_1(e) \int_{-\tilde{x}_1}^{\tilde{x}_1} \left[W_{10}(x) - W_{00} \right] dx && (20)
\end{aligned}
$$

Wages are denoted by w and the instantaneous value in unemployment is denoted by b. They are exogenous and equal for all workers. We assume $w > b$. Recall that $\Psi_1(e) = \omega e \lambda + \psi$ and $\Psi_2(e) = \left[(1 - \omega + \omega e) \lambda + \psi \right]$ are the repective rates at which an unemployed member of a d_0 and d_1 dyad finds a job. Eqs. (17) and (19), which represent the Bellman equation of an employed worker whose partner is employed and unemployed, respectively are rather standard. Eq. (18), represent the Bellman equation of an unemployed worker whose partner is employed and who receives an instantaneous value in unemployment b with a rate to exit unemployment $\Psi_2(e)$ times the gain of becoming employed when searching in the area $2\tilde{x}_2$. Eq. (20) has a similar interpretation, but with job-seekers who leave unemployment at a rate $\Psi_1(e)$ and search for work in the area $2\tilde{x}_1$.

Recall that the maximum commuting distance a job-seeker is willing to accept is denoted by $\tilde{x}_2$ when her strong tie is employed and $\tilde{x}_1$, when her strong tie is unemployed,

[9] See Calvó-Armengol, Verdier, and Zenou (2007) for a similar analysis of employment versus crime.

therefore defined as a job search area. These thresholds can be interpreted as levels of job-seekers' choosiness. The idea is that workers do not randomly apply for work. Instead, they select the most suitable firms located within an acceptable commuting distance. Job-seekers' choosiness when their strong tie is employed and unemployed are given by the following conditions:

$$W_{11}(\tilde{x}_2) - W_{01} = 0 \tag{21}$$

$$W_{10}(\tilde{x}_1) - W_{00} = 0 \tag{22}$$

Proposition 3. *For* $0 < e < 1$ *there exist a unique steady-state interior threshold* $0 < \tilde{\mathcal{X}}_1(e), \tilde{\mathcal{X}}_2(e) < 1/2$*, which solves Eqs.* (22) *and which* (21)*, that yields:*

$$\Psi_2(e)\tau \cdot \tilde{x}_2^2 + (r+\delta)\,\tau \cdot \tilde{x}_2 - (w-b)\,(r+\delta) \;=\; 0 \tag{23}$$

$$\Psi_1(e)\tau \cdot \tilde{x}_1^2 + (r+\delta)\,\tau \cdot \tilde{x}_1 - (w-b)\,(r+\delta) \;=\; 0 \tag{24}$$

with positive roots

$$\tilde{\mathcal{X}}_2(e) = \frac{-(r+\delta)\,\tau + \sqrt[2]{(r+\delta)^2\,\tau^2 + 4\Psi_2(e)\tau\,(w-b)\,(r+\delta)}}{2\Psi_2(e)\tau} \tag{25}$$

$$\tilde{\mathcal{X}}_1(e) = \frac{-(r+\delta)\,\tau + \sqrt[2]{(r+\delta)^2\,\tau^2 + 4\Psi_1(e)\tau\,(w-b)\,(r+\delta)}}{2\Psi_1(e)\tau} \tag{26}$$

Proof. For the proof see Appendix E and Appendix F. □

The exogenous cost of the job search is arbitrarily small. Still, workers face a loss if they apply for a job that is too far away from their residence because they are likely to reject it. Therefore, workers apply to firms for which they are certain to accept an offer. In other words, they choose a job search area till certain threshold (see Decreuse, 2008, for a similar approach).

Lemma 2. *The job search area of individuals whose strong tie is unemployed is larger than the job search area of individuals whose strong tie is employed, i.e.,* $\tilde{\mathcal{X}}_1(e) > \tilde{\mathcal{X}}_2(e)$.

Proof. By the definition of the global rates of leaving unemployment we have that $\Psi_1(e) < \Psi_2(e)$. From Eqs. (23) and (24) and the properties of an upward parabola, it is easy to show that $\Psi_1(e) < \Psi_2(e)$ implies $\tilde{x}_2 < \tilde{x}_1$. Fig. 3 presents a graph that supports this proof. □

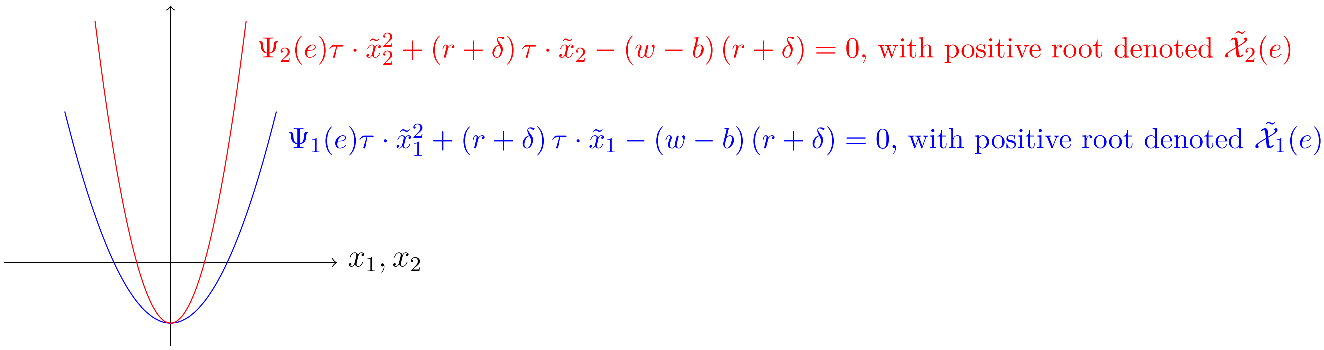

Fig. 3. Job search area for partially forward looking dyads

For those that have an unemployed and employed strong tie, $\tilde{x}_1$ and $\tilde{x}_2$, the higher the employment rate, the smaller the job search area or the choosier the job-seekers. Individuals are more choosy when the probability of meeting a weak tie, ω, increases because they have a higher rate of finding a job and hence reduce their job search areas.

Definition 2. *An equilibrium is 3-tuple* $\{\tilde{e}, \tilde{\mathcal{X}}_1, \tilde{\mathcal{X}}_2\}$ *that verifies the employment rate* (8) *and the job search areas* (25) *and* (26).

Existence and Uniqueness.

The existence and uniqueness of the equilibrium are is determined by means of a fixed-point argument. According to Eq. (11), employment is an increasing function of job search areas $\tilde{\mathcal{X}}_1(e)$ and $\tilde{\mathcal{X}}_2(e)$. In turn, individual choosiness increases with employment. Hence, solving reduces to determine the fixed-point of $\tilde{e} = \tilde{e}\left(\tilde{e}, \mathcal{X}_1(\tilde{e}), \mathcal{X}_2(\tilde{e})\right)$. See Appendix G for computational details.

3. Numerical analysis

We calibrate the model monthly, with $r = 0.0033$. We assume a job destruction rate $\delta = 0.036$ (Pissarides, 2009) and wages $w = 2.5$ and $b = 1.25$ with a replacement rate of $b/w = 0.5$ (Calvó-Armengol et al., 2007). Commuting costs $\tau = 1$, or around 40% of the wage (Zenou, 2009a). We assume that workers spend half of their time with weak ties, $\omega = 0.5$. Considering Assumption 1, the rate at which workers hear about a position is $\lambda = 0.7$ and the rate at which unemployed workers contact firms directly is $\psi = 0.6$. We obtain an employment rate of 97%. The job search area for individuals with an employed strong tie is $\tilde{x}_1 = 0.25$ and the job search area for individuals with an unemployed strong tie is $\tilde{x}_2 = 0.21$. Hence, here we numerically confirm Lemma 2.

Moreover, we observe that the total effect of the time spent with weak ties, ω, is no longer positive in equilibrium. This is because employment is now affected by individual choosiness. An increase of ω has an ambiguous effect on the job search area of individuals whose strong tie is employed, $\tilde{x}_2$; therefore, the effect of ω in employment becomes ambiguous.

4. The labor supply: Perfectly forward looking dyads

In this section, individuals are no longer myopic and can anticipate theirs and their partners' future payoffs (i.e., all agents are perfectly forward looking). In this setup the first part of the Bellman equations is exactly the same as in Eqs. (17)-(20). The last term refers to their dyads either when they exit employment at a rate δ or when they find employment at rates $\Psi_1(e)$ and $\Psi_2(e)$.

$$\begin{aligned}
rW_{11}(x) &= w - \tau \cdot x - \delta \left[W_{11}(x) - W_{01}\right] - \delta \left[W_{11}(x) - W_{10}(x)\right] && (27)\\
rW_{01} &= b + \Psi_2(e) \int_{-\tilde{x}_2}^{\tilde{x}_2} \left[W_{11}(x) - W_{01}\right] dx - \delta \left[W_{01} - W_{00}\right] && (28)\\
rW_{10}(x) &= w - \tau \cdot x - \delta \left[W_{10}(x) - W_{00}\right] + \Psi_2(e) \int_{-\tilde{x}_2}^{\tilde{x}_2} \left[W_{11}(x) - W_{10}(x)\right] dx && (29)\\
rW_{00} &= b + \Psi_1(e) \int_{-\tilde{x}_1}^{\tilde{x}_1} W_{10}(x)\, dx + 2\tilde{x}_1 \Psi_1(e) \left[W_{01} - W_{00}\right] && (30)
\end{aligned}$$

Recall that $\Psi_1(e) = \omega e \lambda + \psi$ is the rate at which an unemployed worker member of a d_0 dyad finds a job, and $\Psi_2(e) = (1 - \omega + \omega e)\, \lambda + \psi$ is the rate at which an unemployed worker member of a d_1 dyad finds a job. The equilibrium conditions are again determined by Eqs. (22) and (21). Solving for the equilibrium using Eqs. (27) to (30) is cumbersome. Therefore, we assume that $\tilde{x}_1 = 1/2$ or the maximum job search area.

Lemma 3. *Assuming $\tilde{x}_1 = 1/2$, solving the system of Eqs.* (27) *to* (30) *for $\tilde{x}_2$ and using the equilibrium condition* (21) *yields the following implicit function for $\tilde{x}_2$.*

$$\beta_5\, \tilde{x}_2^5 + \beta_4\, \tilde{x}_2^4 + \beta_3\, \tilde{x}_2^3 + \beta_2\, \tilde{x}_2^2 + \beta_1\, \tilde{x}_2 + \beta_0 \;=\; 0 \qquad (31)$$

with $\beta_k = \beta_k(\tilde{e})$ for $k = \{0, 1, 2, 3, 4, 5\}$.

Proof. See Appendix H for computational details. □

The solution for the level of choosiness when workers are perfectly forward looking and have an employed strong tie, $\tilde{x}_2$, cannot be solved explicitly as in the case of the partially forward looking individuals. However, we can characterize the equilibrium in the following definition.

Definition 3. *An equilibrium is pair* $\{\tilde{e}_1, \tilde{\mathcal{X}}_2\}$ *that verifies employment* (11) *and the job search area* (31) *for* $0 < \tilde{\mathcal{X}}_2 < 1/2$.

A simple numerical exercise to study the employment in Eq. (31) reveals that there is one positive root $0 < \tilde{\mathcal{X}}_2 < 1/2$, which decreases with employment. Compared to myopic individuals, individuals who consider the future payoffs of their partner and themselves have larger job search areas.

5. Discussion

In this article, we discuss the effect of social networks and the spatial heterogeneity of economic agents on employment. We argue that, because of the spatial heterogeneity of economic agents and given that job-seekers find employment through their social network and the market, they have a different job search area depending on their social network. We model the social network by using dyads (Granovetter, 1973). In our model, individuals have strong and weak ties. We find that myopic individuals with employed partners search for work in a smaller area, compared to individuals whose strong tie is unemployed (i.e., the former are defined as more "choosy"). This interesting effect on employment occurs because the time spent with weak ties no longer increases employment, but rather becomes ambiguous. Similar to the labor network framework in Zenou (2015) we observe that the more time spent with weak ties has a positive impact on employment, because job-seekers are more likely to find a job through meeting employed weak ties. In our case, when spatial dimension enters job-seekers' decision processes, the effect on employment is ambiguous.

This framework can be particularly useful to understand why ethnic minorities experience adverse labor-market outcomes, since the explicit analysis of their networks and the connection to the geographic location is crucial. There is evidence that minorities suffer from a "social and spatial mismatch" (Zenou, 2013) in a monocentric setup. This mismatch has not been studied in more realistic policentric cities, however. Compared to the rest of the population, minorities may have even larger job search areas. In this context, employed members of a minority group may automatically increase the probability that his strong tie will leave unemployment as well.

The model is somewhat cumbersome. Smoothing consumption may solve for the job search areas and compute the expected utilities of each worker, assuming perfect capital markets with zero interest rates. Efficiency is another point in the agenda. In that setup we can study economic policies, such as subsidies to commuting costs for the whole population or only for the unemployed (i.e., dyads in d_0 and d_1 states). A subsidy to the rate at which unemployed workers directly contact firms could be understood as an employment agency subsidized by the social planner.

6. Conclusion

We attempt to determine Why social networks and commuting distance are important to the study of (un)employment rates. We show that the probability of finding a job increases for one partner in a dyad when theother partner gets a job. Moreover, the status of the dyad changes as the strong tie passes from unemployed-unemployed, for instance, to unemployed employed. Hence we observe a "social multiplier." or a propagation effect.

This paper provides a framework where the workforce lives on the circumference of a circle. It is a stylized representation of multicentric cities. Individuals direct their job-search. Job-seekers with an employed strong tie search in smaller range of markets than do job-seekers with an unemployed strong tie. Thus, job-seekers with an employed strong tie are choosier. This job-seeking behavior affects employment rates, which rise as workers participate in more markets. The effect of a worker's social network on employment rates is ambiguous.

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Appendix

A. Proof of Lemma 1

Substituting Eqs. (4) and (5) into Eq. (6) we obtain

$$\tilde{d}_0 = \frac{1}{2} \cdot \frac{\delta^2}{\delta^2 + 4\tilde{x}_1\Psi_1(\tilde{e}) \cdot [\tilde{x}_2\Psi_2(\tilde{e}) + \delta]} \tag{A.1}$$

Substituting Eqs. (4), (5) and (A.1) into Eq. (8) we obtain

$$\tilde{e} = \frac{2\tilde{x}_1\Psi_1(\tilde{e}) \cdot [2\tilde{x}_2\Psi_2(\tilde{e}) + \delta]}{\delta^2 + 4\tilde{x}_1\Psi_1(\tilde{e}) \cdot [\tilde{x}_2\Psi_2(\tilde{e}) + \delta]} \tag{A.2}$$

which writes

$$\tilde{e}\delta^2 + 4\tilde{x}_1\tilde{e}\Psi_1(\tilde{e}) \cdot [\tilde{x}_2\Psi_2(\tilde{e}) + \delta] = 2\tilde{x}_1\Psi_1(\tilde{e}) \cdot [2\tilde{x}_2\Psi_2(\tilde{e}) + \delta] \tag{A.3}$$

Substituting $\Psi_1(\tilde{e}) = \omega\lambda\tilde{e} + \psi$ and $\Psi_2(\tilde{e}) = (1 - \omega + \omega\tilde{e})\,\lambda + \psi$ in Eq. (A.3) yields

$$\tilde{e}\delta^2 + 4\tilde{x}_1\tilde{e}\,(\omega\lambda\tilde{e} + \psi)\,\{\tilde{x}_2\,[(1 - \omega + \omega\tilde{e})\,\lambda + \psi] + \delta\} = 2\tilde{x}_1\,(\omega\lambda\tilde{e} + \psi)\,\{2\tilde{x}_2\,[(1 - \omega + \omega\tilde{e})\,\lambda + \psi] + \delta\}$$

After some computations we obtain:

$$\begin{aligned} &4\tilde{x}_1\tilde{x}_2\omega^2\lambda^2\tilde{e}^3 + 4\tilde{x}_1\omega\lambda\tilde{e}^2\,\{\tilde{x}_2\,[(1-\omega)\,\lambda + \psi] + \delta + \tilde{x}_2\psi - \tilde{x}_2\omega\lambda\} \\ &+\tilde{e}\left\langle \delta^2 + 4\tilde{x}_1\psi\,\{\tilde{x}_2\,[(1-\omega)\,\lambda + \psi] + \delta\} - 2\tilde{x}_1\omega\lambda\,\{2\tilde{x}_2\,[(1-\omega)\,\lambda + \psi] + \delta + 2\tilde{x}_2\psi\}\right\rangle \\ &-2\tilde{x}_1\psi\,\{2\tilde{x}_2\,[(1-\omega)\,\lambda + \psi] + \delta\} = 0 \end{aligned} \tag{A.4}$$

Let define

$$\begin{aligned} \alpha_3 &= 4\tilde{x}_1\tilde{x}_2\omega^2\lambda^2 \\ \alpha_2 &= 4\tilde{x}_1\omega\lambda\,\{\tilde{x}_2\,[(1-\omega)\,\lambda + \psi] + \delta + \tilde{x}_2\psi - \tilde{x}_2\omega\lambda\} \\ \alpha_1 &= \delta^2 + 4\tilde{x}_1\psi\,\{\tilde{x}_2\,[(1-\omega)\,\lambda + \psi] + \delta\} - 2\tilde{x}_1\omega\lambda\,\{2\tilde{x}_2\,[(1-\omega)\,\lambda + \psi] + \delta + 2\tilde{x}_2\psi\} \\ \alpha_0 &= 2\tilde{x}_1\psi\,\{2\tilde{x}_2\,[(1-\omega)\,\lambda + \psi] + \delta\} \end{aligned}$$

Thus, Eq. (A.4) writes

$$\mathcal{E}\,(\tilde{e}) = \alpha_3\tilde{e}^3 + \alpha_2\tilde{e}^2 + \alpha_1\tilde{e} - \alpha_0 = 0$$

We check next if α_3, α_2, α_1 and α_0 are strictly positive.

$$\alpha_3 = 4\tilde{x}_1\tilde{x}_2\omega^2\lambda^2 > 0 \text{ because } 0 < \tilde{x}_{1,}\tilde{x}_2 < 1/2 \text{ and } \omega, \lambda > 0$$

$$\alpha_2 = 4\tilde{x}_1\omega\lambda\left\{\tilde{x}_2\left[(1-\omega)\lambda+\psi\right]+\delta+\tilde{x}_2\psi-\tilde{x}_2\omega\lambda\right\}$$
$$= 4\tilde{x}_1\omega\lambda\left\{\tilde{x}_2\left[(1-\omega)\lambda+\psi\right]+\delta+\tilde{x}_2\left(\psi-\omega\lambda\right)\right\}$$

Hence, $\psi-\omega\lambda>0$ or $\dfrac{\psi}{\lambda}\geqslant\omega$ which is the condition for $b>0$

$$\alpha_1 = \delta^2+4\tilde{x}_1\psi\left\{\tilde{x}_2\left[(1-\omega)\lambda+\psi\right]+\delta\right\}-2\tilde{x}_1\omega\lambda\left\{2\tilde{x}_2\left[(1-\omega)\lambda+\psi\right]+\delta+2\tilde{x}_2\psi\right\}$$
$$=\delta^2+4\tilde{x}_1\tilde{x}_2\psi\left[(1-\omega)\lambda+\psi\right]+4\tilde{x}_1\psi\delta-4\tilde{x}_1\tilde{x}_2\omega\lambda\left[(1-\omega)\lambda+\psi\right]-2\tilde{x}_1\omega\lambda\delta-4\tilde{x}_1\tilde{x}_2\omega\lambda\psi$$
$$=4\tilde{x}_1\tilde{x}_2\left\{\left[(1-\omega)\lambda+\psi\right](\psi-\omega\lambda)-\omega\lambda\psi+\psi^2-\psi^2\right\}+\delta\left(\delta+4\tilde{x}_1\psi-2\tilde{x}_1\omega\lambda\right)$$
$$=4\tilde{x}_1\tilde{x}_2\left\{\left[(1-\omega)\lambda+\psi\right](\psi-\omega\lambda)+\psi\left(\psi-\omega\lambda\right)-\psi^2\right\}+\delta\left[\delta+2\tilde{x}_1\left(2\psi-\omega\lambda\right)\right]$$
$$=4\tilde{x}_1\tilde{x}_2\left\{(\psi-\omega\lambda)\left[(1-\omega)\lambda+2\psi\right]-\psi^2\right\}+\delta\left[\delta+2\tilde{x}_1\left(2\psi-\omega\lambda\right)\right]$$

Hence, we need two conditions for $c>0$

$\psi-\omega\lambda>0$ or $\dfrac{\psi}{\lambda}\geqslant\omega$ and $2\psi-\omega\lambda>0$ or $\dfrac{\psi}{\lambda}\geqslant\dfrac{\omega}{2}$

$$\alpha_0 = 2\tilde{x}_1\psi\left\{2\tilde{x}_2\left[(1-\omega)\lambda+\psi\right]+\delta\right\}>0$$

B. Proof of Proposition 1

By the Decartes' sign rule we know that there is one real positive root and two real negative roots or two imaginary roots. Now we check if the positive root is between zero and one. For $\tilde{e}=0$ we obtain $\mathcal{E}(0)<0$

$$\mathcal{E}(0) = -\alpha_0 < 0$$

For $\tilde{e}=1$ we obtain $\mathcal{E}(1)>0$

$$\mathcal{E}(1) = \alpha_3+\alpha_2+\alpha_1-\alpha_0$$

Substituting the value of α_3, α_2, α_1 and α_0 we obtain

$$\begin{aligned}\mathcal{E}(1) =& 4\tilde{x}_1\tilde{x}_2\omega^2\lambda^2+4\tilde{x}_1\omega\lambda\left\{\tilde{x}_2\left[(1-\omega)\lambda+\psi\right]+\delta+\tilde{x}_2\psi-\tilde{x}_2\omega\lambda\right\}+\delta^2\\ &+4\tilde{x}_1\psi\left\{\tilde{x}_2\left[(1-\omega)\lambda+\psi\right]+\delta\right\}-2\tilde{x}_1\omega\lambda\left\{2\tilde{x}_2\left[(1-\omega)\lambda+\psi\right]+\delta+2\tilde{x}_2\psi\right\}\\ &-2\tilde{x}_1\psi\left\{2\tilde{x}_2\left[(1-\omega)\lambda+\psi\right]+\delta\right\}\\ \mathcal{E}(1) =& 4\tilde{x}_1\tilde{x}_2\omega^2\lambda^2+4\tilde{x}_1\tilde{x}_2\omega\lambda\left[(1-\omega)\lambda+\psi\right]+4\tilde{x}_1\omega\lambda\delta+4\tilde{x}_1\tilde{x}_2\omega\lambda\psi-4\tilde{x}_1\tilde{x}_2\omega^2\lambda^2+\delta^2\\ &+4\tilde{x}_1\tilde{x}_2\psi\left[(1-\omega)\lambda+\psi\right]+4\tilde{x}_1\psi\delta-4\tilde{x}_1\tilde{x}_2\omega\lambda\left[(1-\omega)\lambda+\psi\right]\end{aligned}$$

$$- 2\tilde{x}_1\omega\lambda\delta - 4\tilde{x}_1\tilde{x}_2\omega\lambda\psi - 4\tilde{x}_1\tilde{x}_2\psi\left[(1-\omega)\lambda + \psi\right] - 2\tilde{x}_1\psi\delta$$

$$\mathcal{E}(1) = \delta\left(\delta + 2\tilde{x}_1\psi + 2\tilde{x}_1\omega\lambda\right)$$

$$\mathcal{E}(1) = \delta + 2\tilde{x}_1\psi + 2\tilde{x}_1\omega\lambda > 0$$

Which proves that the only positive real root $\tilde{e}$ is between zero and one, $0 < \tilde{e} < 1$.

C. Proof of Proposition 2

First we divide (9) by α_3 and obtain:

$$\mathcal{E}(e) = \tilde{e}^3 + a_2\tilde{e}^2 + a_1\tilde{e} - a_0 = 0 \tag{C.1}$$

where:

$$\begin{aligned}
a_2 &= \frac{1}{\omega\lambda}\left[(1-\omega)\lambda + (2\psi - \omega\lambda) + \frac{\delta}{\tilde{x}_2}\right] \\
a_1 &= \frac{1}{\omega^2\lambda^2}\left[\frac{\delta^2}{4\tilde{x}_1\tilde{x}_2} + (1-\omega)\lambda(\psi - \omega\lambda) + \psi(\psi - 2\omega\lambda) + \frac{\delta(2\psi - \omega\lambda)}{2\tilde{x}_2}\right] \\
a_0 &= \frac{1}{\omega^2\lambda^2}\left\{\psi\left[(1-\omega)\lambda + \psi\right] + \frac{\psi\delta}{2\tilde{x}_2}\right\}
\end{aligned}$$

We reduce the cubic function (C.1) by the following substitution: $\tilde{e} = z - \frac{a_2}{3}$ to the normal form:

$$z^3 + c_1 z + c_0 = 0 \tag{C.2}$$

where

$$\begin{aligned}
c_1 &= \frac{1}{3}\left[3a_1 - a_2^2\right] \\
c_0 &= \frac{1}{27}\left(2a_2^3 - 9a_1a_2 - 27a_0\right)
\end{aligned}$$

We use the formula of the discriminant to find out more about the roots. We apply the formula of the discriminant to the reduced cubic function (C.2) and we find the following equation:

$$\Delta = -(4c_1^3 + 27c_0^2) > 0 \text{ since } c_1 < 0, c_0 < 0. \tag{C.3}$$

Since the cubic function (C.2) has a positive discriminant, it has three distinct real roots. We use next the trigonometric approach to define the roots of (C.2).

$$\begin{aligned}
z_1 &= 2g\cos\theta \Longrightarrow \tilde{e}_1 = 2g\cos\theta - \frac{a_2}{3} > 0 \\
z_2 &= 2g\cos\theta \Longrightarrow \tilde{e}_2 = 2g\cos\left(\frac{2\pi}{3} - \theta\right) - \frac{a_2}{3} < 0 \\
z_3 &= 2g\cos\theta \Longrightarrow \tilde{e}_3 = 2g\cos\left(\frac{2\pi}{3} + \theta\right) - \frac{a_2}{3} < 0
\end{aligned}$$

where

$$\begin{aligned} g &= \sqrt[2]{p} \text{ where } p = \frac{-c_1}{3} \\ h &= 2g^3 \\ \theta &= \frac{1}{3}\cos^{-1}(q) \text{ where } q = \frac{-c_0}{h} \end{aligned}$$

D. Comparative statics of the dyads

Substituting Eq. (G.1) into Eq. (4) yields

$$\tilde{d}_2 = \frac{2\tilde{x}_1\tilde{x}_2\Psi_1(\tilde{e})\Psi_2(\tilde{e})}{\delta^2 + 4\tilde{x}_1\Psi_1(\tilde{e}) \cdot [\tilde{x}_2\Psi_2(\tilde{e}) + \delta]} \tag{D.1}$$

Evaluating Eq. (G.8)

- $\frac{\partial \tilde{d}_2}{\partial \tilde{x}_1} = \frac{2\tilde{x}_2\Psi_1(\tilde{e})\Psi_2(\tilde{e})\delta^2}{[\delta^2+4\tilde{x}_1\Psi_1(\tilde{e})\cdot[\tilde{x}_2\Psi_2(\tilde{e})+\delta]]^2} > 0$
- $\frac{\partial \tilde{d}_2}{\partial \tilde{x}_2} = \frac{2\tilde{x}_1\Psi_1(\tilde{e})\Psi_2(\tilde{e})\delta(\delta+4\tilde{x}_1\Psi_1(\tilde{e}))}{[\delta^2+4\tilde{x}_1\Psi_1(\tilde{e})\cdot[\tilde{x}_2\Psi_2(\tilde{e})+\delta]]^2} > 0$
- $\frac{\partial \tilde{d}_2}{\partial \Psi_1(\tilde{e})} = \frac{2\tilde{x}_1\tilde{x}_2\Psi_2(\tilde{e})\delta^2}{[\delta^2+4\tilde{x}_1\Psi_1(\tilde{e})\cdot[\tilde{x}_2\Psi_2(\tilde{e})+\delta]]^2} > 0$
- $\frac{\partial \tilde{d}_2}{\partial \Psi_2(\tilde{e})} = \frac{2\tilde{x}_1\tilde{x}_2\Psi_1(\tilde{e})\delta(\delta+4\tilde{x}_1\Psi_1(\tilde{e}))}{[\delta^2+4\tilde{x}_1\Psi_1(\tilde{e})\cdot[\tilde{x}_2\Psi_2(\tilde{e})+\delta]]^2} > 0$

By the definition of $\Psi_1(\tilde{e}) = \omega e\lambda + \psi$ and $\Psi_2(\tilde{e}) = (1 - \omega + \omega e)\,\lambda + \psi$

- $\frac{\partial \Psi_1(\tilde{e})}{\partial \tilde{e}} = \frac{\partial \Psi_2(\tilde{e})}{\partial \tilde{e}} = \omega\lambda > 0$
- $\frac{\partial \Psi_1(\tilde{e})}{\partial \psi} = \frac{\partial \Psi_2(\tilde{e})}{\partial \psi} = 1 > 0$
- $\frac{\partial \Psi_1(\tilde{e})}{\partial \lambda} = \omega e > 0$
- $\frac{\partial \Psi_1(\tilde{e})}{\partial \omega} = e\lambda > 0$
- $\frac{\partial \Psi_2(\tilde{e})}{\partial \lambda} = (1 - \omega + \omega e) > 0$
- $\frac{\partial \Psi_2(\tilde{e})}{\partial \omega} = -1 + e\lambda \lesseqgtr 0$

  Hence,

- $\frac{\partial \tilde{d}_2}{\partial \tilde{e}} = \frac{\partial \tilde{d}_2}{\partial \Psi_1(\tilde{e})} \cdot \frac{\partial \Psi_1(\tilde{e})}{\partial \tilde{e}} + \frac{\partial \tilde{d}_2}{\partial \Psi_2(\tilde{e})} \cdot \frac{\partial \Psi_2(\tilde{e})}{\partial \tilde{e}} > 0$
- $\frac{\partial \tilde{d}_2}{\partial \psi} = \frac{\partial \tilde{d}_2}{\partial \Psi_1(\tilde{e})} \cdot \frac{\partial \Psi_1(\tilde{e})}{\partial \psi} + \frac{\partial \tilde{d}_2}{\partial \Psi_2(\tilde{e})} \cdot \frac{\partial \Psi_2(\tilde{e})}{\partial \psi} > 0$
- $\frac{\partial \tilde{d}_2}{\partial \lambda} = \frac{\partial \tilde{d}_2}{\partial \Psi_1(\tilde{e})} \cdot \frac{\partial \Psi_1(\tilde{e})}{\partial \lambda} + \frac{\partial \tilde{d}_2}{\partial \Psi_2(\tilde{e})} \cdot \frac{\partial \Psi_2(\tilde{e})}{\partial \lambda} > 0$

- $\frac{\partial \tilde{d}_2}{\partial \omega} = \frac{\partial \tilde{d}_2}{\partial \Psi_1(\tilde{e})} \cdot \frac{\partial \Psi_1(\tilde{e})}{\partial \omega} + \frac{\partial \tilde{d}_2}{\partial \Psi_2(\tilde{e})} \cdot \frac{\partial \Psi_2(\tilde{e})}{\partial \omega} \lesseqgtr 0$

Substituting Eq. (G.1) into Eq. (5) yields

$$\tilde{d}_1 = \frac{2\tilde{x}_1\Psi_1(\tilde{e})\delta}{\delta^2 + 4\tilde{x}_1\Psi_1(\tilde{e}) \cdot [\tilde{x}_2\Psi_2(\tilde{e}) + \delta]} \tag{D.2}$$

Evaluating Eq. (G.9)

- $\frac{\partial \tilde{d}_1}{\partial \tilde{x}_1} = \frac{2\Psi_1(\tilde{e})\delta^3}{[\delta^2+4\tilde{x}_1\Psi_1(\tilde{e})\cdot[\tilde{x}_2\Psi_2(\tilde{e})+\delta]]^2} > 0$
- $\frac{\partial \tilde{d}_1}{\partial \tilde{x}_2} = -\frac{8\tilde{x}_1^2\Psi_1^2(\tilde{e})\Psi_2(\tilde{e})\delta}{[\delta^2+4\tilde{x}_1\Psi_1(\tilde{e})\cdot[\tilde{x}_2\Psi_2(\tilde{e})+\delta]]^2} < 0$
- $\frac{\partial \tilde{d}_1}{\partial \Psi_1(\tilde{e})} = \frac{2\tilde{x}_1\delta^3}{[\delta^2+4\tilde{x}_1\Psi_1(\tilde{e})\cdot[\tilde{x}_2\Psi_2(\tilde{e})+\delta]]^2} > 0$
- $\frac{\partial \tilde{d}_1}{\partial \Psi_2(\tilde{e})} = -\frac{8\tilde{x}_1^2\tilde{x}_2\Psi_1^2(\tilde{e})\delta}{[\delta^2+4\tilde{x}_1\Psi_1(\tilde{e})\cdot[\tilde{x}_2\Psi_2(\tilde{e})+\delta]]^2} < 0$
- $\frac{\partial \tilde{d}_1}{\partial \tilde{e}} = \frac{\partial \tilde{d}_1}{\partial \Psi_1(\tilde{e})} \cdot \frac{\partial \Psi_1(\tilde{e})}{\partial \tilde{e}} + \frac{\partial \tilde{d}_1}{\partial \Psi_2(\tilde{e})} \cdot \frac{\partial \Psi_2(\tilde{e})}{\partial \tilde{e}} = -\omega\lambda\frac{2\tilde{x}_1\delta(\delta-4\tilde{x}_1\tilde{x}_2\Psi_2^2(\tilde{e})\delta)}{[\delta^2+4\tilde{x}_1\Psi_1(\tilde{e})\cdot[\tilde{x}_2\Psi_2(\tilde{e})+\delta]]^2} < 0$
- $\frac{\partial \tilde{d}_1}{\partial \psi} = \frac{\partial \tilde{d}_1}{\partial \Psi_1(\tilde{e})} \cdot \frac{\partial \Psi_1(\tilde{e})}{\partial \psi} + \frac{\partial \tilde{d}_1}{\partial \Psi_2(\tilde{e})} \cdot \frac{\partial \Psi_2(\tilde{e})}{\partial \psi} = -\frac{2\tilde{x}_1\delta(\delta-4\tilde{x}_1\tilde{x}_2\Psi_2^2(\tilde{e})\delta)}{[\delta^2+4\tilde{x}_1\Psi_1(\tilde{e})\cdot[\tilde{x}_2\Psi_2(\tilde{e})+\delta]]^2} < 0$
- $\frac{\partial \tilde{d}_1}{\partial \lambda} = \frac{\partial \tilde{d}_1}{\partial \Psi_1(\tilde{e})} \cdot \frac{\partial \Psi_1(\tilde{e})}{\partial \lambda} + \frac{\partial \tilde{d}_1}{\partial \Psi_2(\tilde{e})} \cdot \frac{\partial \Psi_2(\tilde{e})}{\partial \lambda} \lesseqgtr 0$
- $\frac{\partial \tilde{d}_1}{\partial \omega} = \frac{\partial \tilde{d}_1}{\partial \Psi_1(\tilde{e})} \cdot \frac{\partial \Psi_1(\tilde{e})}{\partial \omega} + \frac{\partial \tilde{d}_1}{\partial \Psi_2(\tilde{e})} \cdot \frac{\partial \Psi_2(\tilde{e})}{\partial \omega} \lesseqgtr 0$

Finally, we evaluate Eq. (G.1)

$$\tilde{d}_0 = \frac{1}{2} \cdot \frac{\delta^2}{\delta^2 + 4\tilde{x}_1\Psi_1(\tilde{e}) \cdot [\tilde{x}_2\Psi_2(\tilde{e}) + \delta]}. \tag{D.3}$$

- $\frac{\partial \tilde{d}_0}{\partial \tilde{x}_1} = -\frac{4\Psi_1(\tilde{e})\cdot[\tilde{x}_2\Psi_2(\tilde{e})+\delta]\delta^2}{4[\delta^2+4\tilde{x}_1\Psi_1(\tilde{e})\cdot[\tilde{x}_2\Psi_2(\tilde{e})+\delta]]^2} < 0$
- $\frac{\partial \tilde{d}_0}{\partial \tilde{x}_2} = -\frac{4\tilde{x}_1\Psi_1(\tilde{e})\Psi_2(\tilde{e})\delta^2}{4[\delta^2+4\tilde{x}_1\Psi_1(\tilde{e})\cdot[\tilde{x}_2\Psi_2(\tilde{e})+\delta]]^2} < 0$
- $\frac{\partial \tilde{d}_0}{\partial \Psi_1(\tilde{e})} = -\frac{4\tilde{x}_1\tilde{x}_2\Psi_2(\tilde{e})\delta^2}{4[\delta^2+4\tilde{x}_1\Psi_1(\tilde{e})\cdot[\tilde{x}_2\Psi_2(\tilde{e})+\delta]]^2} < 0$
- $\frac{\partial \tilde{d}_0}{\partial \Psi_2(\tilde{e})} = -\frac{4\tilde{x}_1\tilde{x}_2\Psi_1(\tilde{e})\delta^2}{4[\delta^2+4\tilde{x}_1\Psi_1(\tilde{e})\cdot[\tilde{x}_2\Psi_2(\tilde{e})+\delta]]^2} < 0$
- $\frac{\partial \tilde{d}_0}{\partial \tilde{e}} = \frac{\partial \tilde{d}_1}{\partial \Psi_1(\tilde{e})} \cdot \frac{\partial \Psi_1(\tilde{e})}{\partial \tilde{e}} + \frac{\partial \tilde{d}_1}{\partial \Psi_2(\tilde{e})} \cdot \frac{\partial \Psi_2(\tilde{e})}{\partial \tilde{e}} < 0$
- $\frac{\partial \tilde{d}_0}{\partial \psi} = \frac{\partial \tilde{d}_1}{\partial \Psi_1(\tilde{e})} \cdot \frac{\partial \Psi_1(\tilde{e})}{\partial \psi} + \frac{\partial \tilde{d}_1}{\partial \Psi_2(\tilde{e})} \cdot \frac{\partial \Psi_2(\tilde{e})}{\partial \psi} < 0$
- $\frac{\partial \tilde{d}_0}{\partial \lambda} = \frac{\partial \tilde{d}_1}{\partial \Psi_1(\tilde{e})} \cdot \frac{\partial \Psi_1(\tilde{e})}{\partial \lambda} + \frac{\partial \tilde{d}_1}{\partial \Psi_2(\tilde{e})} \cdot \frac{\partial \Psi_2(\tilde{e})}{\partial \lambda} < 0$
- $\frac{\partial \tilde{d}_0}{\partial \omega} = \frac{\partial \tilde{d}_1}{\partial \Psi_1(\tilde{e})} \cdot \frac{\partial \Psi_1(\tilde{e})}{\partial \omega} + \frac{\partial \tilde{d}_1}{\partial \Psi_2(\tilde{e})} \cdot \frac{\partial \Psi_2(\tilde{e})}{\partial \omega} \lesseqgtr 0$

E. Partially forward looking dyads: Equilibrium $\tilde{x}_2$

The equilibrium condition is given by $W_{11}(\tilde{x}_2) - W_{01} = 0 \Rightarrow \tilde{x}_2$. We evaluate Eq. (17) at $\tilde{x}_2$ and establish the equilibrium condition:

$$
\begin{aligned}
(r+\delta)\left[W_{11}(\tilde{x}_2) - W_{01}\right] &= w - \tau \cdot \tilde{x}_2 - rW_{01} \\
rW_{01} &= w - \tau \cdot \tilde{x}_2
\end{aligned}
\tag{E.1}
$$

Isolate W_{01} from Eq. (18) we obtain

$$
\begin{aligned}
rW_{01} &= b + \Psi_2(e)\int_{-\tilde{x}_2}^{\tilde{x}_2}\left[W_{11}(x) - W_{01}\right]dx \\
rW_{01} &= b + 2\Psi_2(e)\left[\int_0^{\tilde{x}_2} W_{11}(x)\,dx - \int_0^{\tilde{x}_2} W_{01}dx\right] \\
rW_{01} &= b + 2\Psi_2(e)\left[\int_0^{\tilde{x}_2} W_{11}(x)\,dx - \tilde{x}_2 W_{01}\right] \\
\left(r + 2\tilde{x}_2\Psi_2(e)\right)W_{01} &= b + 2\Psi_2(e)\int_0^{\tilde{x}_2} W_{11}(x)\,dx
\end{aligned}
\tag{E.2}
$$

Next, by using the linearity of Eq. (17) we obtain

$$
\begin{aligned}
\frac{1}{\tilde{x}_2}\int_0^{\tilde{x}_2} W_{11}(x)\,dx &= \mathbb{E}_x\left[W_{11}(x)\,|x < \tilde{x}_2\right] \\
&= W_{11}\left[\mathbb{E}_x\left(x|x < \tilde{x}_2\right)\right] \\
&= W_{11}\left(\frac{\tilde{x}_2}{2}\right) \text{ because } \mathbb{E}_x\left(x|x < \tilde{x}_2\right) = \frac{\tilde{x}_2}{2} \\
\int_0^{\tilde{x}_2} W_{11}(x)\,dx &= \tilde{x}_2 W_{11}\left(\frac{\tilde{x}_2}{2}\right)
\end{aligned}
\tag{E.3}
$$

Substituting Eq. (E.3) into Eq. (E.2) we obtain

$$
\left(r + 2\tilde{x}_2\Psi_2(e)\right)W_{01} = b + 2\tilde{x}_2\Psi_2(e)W_{11}\left(\frac{\tilde{x}_2}{2}\right) \tag{E.4}
$$

Evaluating Eq. (18) at $\tilde{x}_2/2$ and isolating $W_{11}(\tilde{x}_2/2)$ yields

$$
(r+\delta)\,W_{11}(\tilde{x}_2/2) = w - \tau \cdot \tilde{x}_2/2 + \delta W_{01} \tag{E.5}
$$

Substituting Eq. (E.5) into Eq. (E.4) yields

$$
\left(r + 2\tilde{x}_2\Psi_2(e)\right)W_{01} = b + 2\tilde{x}_2\Psi_2(e)\frac{w - \tau \cdot \tilde{x}_2/2 + \delta W_{01}}{r+\delta} \tag{E.6}
$$

Isolating W_{01} from Eq. (E.6)

$$(r + 2\tilde{x}_2\Psi_2(e))\,W_{01} = b + \frac{2\tilde{x}_2\Psi_2(e)}{r+\delta}\,(w - \tau\cdot\tilde{x}_2/2) + \frac{2\tilde{x}_2\Psi_2(e)\delta}{r+\delta}W_{01}$$
$$\left[\frac{(r + 2\tilde{x}_2\Psi_2(e))\,(r+\delta) - 2\tilde{x}_2\Psi_2(e)\delta}{r+\delta}\right]W_{01} = \frac{b\,(r+\delta) + 2\tilde{x}_2\Psi_2(e)\,(w - \tau\cdot\tilde{x}_2/2)}{r+\delta}$$
$$[r\,(r+\delta) + 2\tilde{x}_2\Psi_2(e)\,(r+\delta) - 2\tilde{x}_2\Psi_2(e)\delta]\,W_{01} = b\,(r+\delta) + 2\tilde{x}_2\Psi_2(e)\,(w - \tau\cdot\tilde{x}_2/2)$$
$$[r\,(r+\delta) + 2\tilde{x}_2\Psi_2(e)r + 2\tilde{x}_2\Psi_2(e)\delta - 2\tilde{x}_2\Psi_2(e)\delta]\,W_{01} = b\,(r+\delta) + 2\tilde{x}_2\Psi_2(e)\,(w - \tau\cdot\tilde{x}_2/2)$$
$$r\,[r + \delta + 2\tilde{x}_2\Psi_2(e)]\,W_{01} = b\,(r+\delta) + 2\tilde{x}_2\Psi_2(e)\,(w - \tau\cdot\tilde{x}_2/2)$$
$$rW_{01} = \frac{b\,(r+\delta) + 2\tilde{x}_2\Psi_2(e)\,(w - \tau\cdot\tilde{x}_2/2)}{r+\delta+2\tilde{x}_2\Psi_2(e)} \tag{E.7}$$

Substituting Eq. (E.7) into Eq. (E.1) we get

$$\frac{b\,(r+\delta) + 2\tilde{x}_2\Psi_2(e)\,(w - \tau\cdot\tilde{x}_2/2)}{r+\delta+2\tilde{x}_2\Psi_2(e)} = w - \tau\cdot\tilde{x}_2$$
$$b\,(r+\delta) + 2\tilde{x}_2\Psi_2(e)\,(w - \tau\cdot\tilde{x}_2/2) = (w - \tau\cdot\tilde{x}_2)\,(r+\delta+2\tilde{x}_2\Psi_2(e))$$
$$b\,(r+\delta) + 2\tilde{x}_2\Psi_2(e)\,(w - \tau\cdot\tilde{x}_2/2) = w\,(r+\delta+2\tilde{x}_2\Psi_2(e)) - \tau\cdot\tilde{x}_2\,(r+\delta+2\tilde{x}_2\Psi_2(e))$$
$$b\,(r+\delta) + 2\tilde{x}_2\Psi_2(e)w - 2\tilde{x}_2\Psi_2(e)\tau\cdot\tilde{x}_2/2 = w\,(r+\delta) + 2\tilde{x}_2\Psi_2(e)w - (r+\delta)\,\tau\cdot\tilde{x}_2 - 2\Psi_2(e)\tau\cdot\tilde{x}_2^2$$
$$b\,(r+\delta) - \Psi_2(e)\tau\cdot\tilde{x}_2^2 = w\,(r+\delta) - (r+\delta)\,\tau\cdot\tilde{x}_2 - 2\Psi_2(e)\tau\cdot\tilde{x}_2^2$$
$$\Psi_2(e)\tau\cdot\tilde{x}_2^2 + (r+\delta)\,\tau\cdot\tilde{x}_2 - (w-b)\,(r+\delta) = 0 \tag{E.8}$$

The quadratic implicit function (E.8) has one positive and one negative root. The positive root is:

$$\tilde{x}_2 = \frac{-(r+\delta)\,\tau + \sqrt[2]{(r+\delta)^2\,\tau^2 + 4\Psi_2(e)\tau\,(w-b)\,(r+\delta)}}{2\Psi_2(e)\tau} \tag{E.9}$$

F. Partially forward looking dyads: Equilibrium $\tilde{x}_1$

The equilibrium condition is given by $W_{10}\,(\tilde{x}_1) - W_{00} = 0 \Rightarrow \tilde{x}_1$. We evaluate Eq. (19) at $\tilde{x}_1$ and establish the equilibrium condition:

$$(r+\delta)\,[W_{10}\,(\tilde{x}_1) - W_{00}] = w - \tau\cdot\tilde{x}_1 - rW_{00}$$
$$rW_{00} = w - \tau\cdot\tilde{x}_1 \tag{F.1}$$

We study the Bellman Eq. (20) and isolate W_{00}

$$
\begin{aligned}
rW_{00} &= b + \Psi_1(e) \int_{-\tilde{x}_1}^{\tilde{x}_1} \left[W_{10}\left(x\right) - W_{00}\right] dx \\
rW_{00} &= b + 2\Psi_1(e) \left[\int_0^{\tilde{x}_1} W_{10}\left(x\right) dx - \int_0^{\tilde{x}_1} W_{00} dx\right] \\
rW_{00} &= b + 2\Psi_1(e) \left[\int_0^{\tilde{x}_1} W_{10}\left(x\right) dx - \tilde{x}_1 W_{00}\right] \\
\left(r + 2\tilde{x}_1\Psi_1(e)\right) W_{00} &= b + 2\Psi_1(e) \int_0^{\tilde{x}_1} W_{10}\left(x\right) dx \qquad \text{(F.2)}
\end{aligned}
$$

Next, by using the linearity of Eq. (19) we obtain

$$
\begin{aligned}
\frac{1}{\tilde{x}_1} \int_0^{\tilde{x}_1} W_{10}\left(x\right) dx &= \mathbb{E}_x \left[W_{10}\left(x\right) | x < \tilde{x}_1\right] \\
&= W_{10}\left[\mathbb{E}_x\left(x | x < \tilde{x}_1\right)\right] \\
&= W_{10}\left(\frac{\tilde{x}_1}{2}\right) \text{ because } \mathbb{E}_x\left(x | x < \tilde{x}_1\right) = \frac{\tilde{x}_1}{2} \\
\int_0^{\tilde{x}_1} W_{10}\left(x\right) dx &= \tilde{x}_1 W_{10}\left(\frac{\tilde{x}_1}{2}\right) \qquad \text{(F.3)}
\end{aligned}
$$

Substituting Eq. (F.3) into Eq. (F.2) we obtain

$$
\left(r + 2\tilde{x}_1\Psi_1(e)\right) W_{00} = b + 2\tilde{x}_1\Psi_1(e) W_{10}\left(\frac{\tilde{x}_1}{2}\right) \qquad \text{(F.4)}
$$

Evaluating Eq. (19) at $\tilde{x}_1/2$ and isolating $W_{10}\left(\tilde{x}_1/2\right)$ yields

$$
\left(r + \delta\right) W_{10}\left(\tilde{x}_1/2\right) = w - \tau \cdot \tilde{x}_1/2 + \delta W_{00} \qquad \text{(F.5)}
$$

Substituting Eq. (F.5) into Eq. (F.4) yields

$$
\left(r + 2\tilde{x}_1\Psi_1(e)\right) W_{00} = b + 2\tilde{x}_1\Psi_1(e) \frac{w - \tau \cdot \tilde{x}_1/2 + \delta W_{00}}{r + \delta} \qquad \text{(F.6)}
$$

Isolating W_{00} from Eq. (F.6)

$$
\begin{aligned}
\left(r + 2\tilde{x}_1\Psi_1(e)\right) W_{00} &= b + \frac{2\tilde{x}_1\Psi_1(e)}{r + \delta}\left(w - \tau \cdot \tilde{x}_1/2\right) + \frac{2\tilde{x}_1\Psi_1(e)\delta}{r + \delta} W_{00} \\
\left[\frac{\left(r + 2\tilde{x}_1\Psi_1(e)\right)\left(r + \delta\right) - 2\tilde{x}_1\Psi_1(e)\delta}{r + \delta}\right] W_{00} &= \frac{b\left(r + \delta\right) + 2\tilde{x}_1\Psi_1(e)\left(w - \tau \cdot \tilde{x}_1/2\right)}{r + \delta} \\
\left[r\left(r + \delta\right) + 2\tilde{x}_1\Psi_1(e)\left(r + \delta\right) - 2\tilde{x}_1\Psi_1(e)\delta\right] W_{00} &= b\left(r + \delta\right) + 2\tilde{x}_1\Psi_1(e)\left(w - \tau \cdot \tilde{x}_1/2\right) \\
\left[r\left(r + \delta\right) + 2\tilde{x}_1\Psi_1(e) r + 2\tilde{x}_1\Psi_2(e)\delta - 2\tilde{x}_1\Psi_1(e)\delta\right] W_{00} &= b\left(r + \delta\right) + 2\tilde{x}_1\Psi_1(e)\left(w - \tau \cdot \tilde{x}_1/2\right) \\
r\left[r + \delta + 2\tilde{x}_1\Psi_2(e)\right] W_{00} &= b\left(r + \delta\right) + 2\tilde{x}_1\Psi_1(e)\left(w - \tau \cdot \tilde{x}_1/2\right) \\
rW_{00} &= \frac{b\left(r + \delta\right) + 2\tilde{x}_1\Psi_1(e)\left(w - \tau \cdot \tilde{x}_1/2\right)}{r + \delta + 2\tilde{x}_1\Psi_1(e)} \qquad \text{(F.7)}
\end{aligned}
$$

Substituting Eq. (F.7) into Eq. (F.1) yields

$$\frac{b\,(r+\delta)+2\tilde{x}_1\Psi_1(e)\,(w-\tau\cdot\tilde{x}_1/2)}{r+\delta+2\tilde{x}_1\Psi_1(e)} = w-\tau\cdot\tilde{x}_1$$

$$b\,(r+\delta)+2\tilde{x}_1\Psi_1(e)\,(w-\tau\cdot\tilde{x}_1/2) = (w-\tau\cdot\tilde{x}_1)\,(r+\delta+2\tilde{x}_1\Psi_1(e))$$

$$b\,(r+\delta)+2\tilde{x}_1\Psi_1(e)\,(w-\tau\cdot\tilde{x}_1/2) = w\,(r+\delta+2\tilde{x}_1\Psi_1(e))-\tau\cdot\tilde{x}_1\,(r+\delta+2\tilde{x}_1\Psi_1(e))$$

$$b\,(r+\delta)+2\tilde{x}_1\Psi_1(e)w-2\tilde{x}_1\Psi_1(e)\tau\cdot\tilde{x}_1/2 = w\,(r+\delta)+2\tilde{x}_1\Psi_1(e)w-(r+\delta)\,\tau\cdot\tilde{x}_1-2\Psi_1(e)\tau\cdot\tilde{x}_1^2$$

$$b\,(r+\delta)-\Psi_1(e)\tau\cdot\tilde{x}_1^2 = w\,(r+\delta)-(r+\delta)\,\tau\cdot\tilde{x}_1-2\Psi_1(e)\tau\cdot\tilde{x}_1^2$$

$$\Psi_1(e)\tau\cdot\tilde{x}_1^2+(r+\delta)\,\tau\cdot\tilde{x}_1-(w-b)\,(r+\delta)=0 \tag{F.8}$$

The quadratic implicit function Eq. (F.8) has one positive and one negative root. The positive root is:

$$\tilde{x}_1 = \frac{-(r+\delta)\,\tau+\sqrt[2]{(r+\delta)^2\,\tau^2+4\Psi_1(e)\tau\,(w-b)\,(r+\delta)}}{2\Psi_1(e)\tau} \tag{F.9}$$

G. Equilibrium of imperfectly forward looking dyads

We first find an expression for $\tilde{d}_0$. Substituting Eqs. (4) and (5) into (6), we obtain:

$$\tilde{d}_0 = \frac{1}{2}-\tilde{d}_2-\tilde{d}_1$$

$$\tilde{d}_0 = \frac{1}{2}-\frac{4\tilde{x}_1\tilde{x}_2\Psi_1(\tilde{e})\Psi_2(\tilde{e})}{\delta^2}\tilde{d}_0-\frac{4\tilde{x}_1\Psi_1(\tilde{e})}{\delta}\tilde{d}_0$$

Factorizing $\frac{4\tilde{x}_1\Psi_1(\tilde{e})\tilde{d}_0}{\delta^2}$

$$\tilde{d}_0 = \frac{1}{2}-\frac{4\tilde{x}_1\Psi_1(\tilde{e})}{\delta^2}\left[\tilde{x}_2\Psi_2(\tilde{e})+\delta\right]\tilde{d}_0$$

Dividing by $\tilde{d}_0$

$$1 = \frac{1}{2\tilde{d}_0}-\frac{4\tilde{x}_1\Psi_1(\tilde{e})}{\delta^2}\left[\tilde{x}_2\Psi_2(\tilde{e})+\delta\right]$$

Hence,

$$\frac{1}{2\tilde{d}_0} = 1+\frac{4\tilde{x}_1\Psi_1(\tilde{e})}{\delta^2}\left[\tilde{x}_2\Psi_2(\tilde{e})+\delta\right]$$

$$\frac{1}{2\tilde{d}_0} = \frac{\delta^2+4\tilde{x}_1\Psi_1(\tilde{e})\left[\tilde{x}_2\Psi_2(\tilde{e})+\delta\right]}{\delta^2}$$

$$2\tilde{d}_0 = \frac{\delta^2}{\delta^2 + 4\tilde{x}_1\Psi_1(\tilde{e})\left[\tilde{x}_2\Psi_2(\tilde{e}) + \delta\right]}$$

$$\tilde{d}_0 = \frac{1}{2} \cdot \frac{\delta^2}{\delta^2 + 4\tilde{x}_1\Psi_1(\tilde{e}) \cdot \left[\tilde{x}_2\Psi_2(\tilde{e}) + \delta\right]} \text{ where } \tilde{d}_0 \equiv \tilde{d}_0\left(\Psi_1(\tilde{e}), \Psi_2(\tilde{e}), \tilde{x}_1, \tilde{x}_2\right). \qquad \text{(G.1)}$$

Substituting Eq. (G.1) in Eq. (4) yields

$$\begin{aligned} \tilde{d}_2 &= 4\tilde{x}_1\tilde{x}_2 \cdot \frac{\Psi_1(\tilde{e})\Psi_2(\tilde{e})}{\delta^2} \cdot \frac{1}{2} \cdot \frac{\delta^2}{\delta^2 + 4\tilde{x}_1\Psi_1(\tilde{e}) \cdot \left[\tilde{x}_2\Psi_2(\tilde{e}) + \delta\right]} \\ \tilde{d}_2 &= \frac{2\tilde{x}_1\tilde{x}_2\Psi_1(\tilde{e})\Psi_2(\tilde{e})}{\delta^2 + 4\tilde{x}_1\Psi_1(\tilde{e}) \cdot \left[\tilde{x}_2\Psi_2(\tilde{e}) + \delta\right]} \text{ where } \tilde{d}_2 = \tilde{d}_2\left(\Psi_1(\tilde{e}), \Psi_2(\tilde{e}), \tilde{x}_1, \tilde{x}_2\right) \qquad \text{(G.2)} \end{aligned}$$

Substituting Eq. (G.1) in Eq. (5) yields

$$\begin{aligned} \tilde{d}_1 &= 4\tilde{x}_1 \cdot \frac{\Psi_1(\tilde{e})}{\delta}\frac{1}{2} \cdot \frac{\delta^2}{\delta^2 + 4\tilde{x}_1\Psi_1(\tilde{e}) \cdot \left[\tilde{x}_2\Psi_2(\tilde{e}) + \delta\right]} \\ \tilde{d}_1 &= \frac{2\tilde{x}_1\Psi_1(\tilde{e})\delta}{\delta^2 + 4\tilde{x}_1\Psi_1(\tilde{e}) \cdot \left[\tilde{x}_2\Psi_2(\tilde{e}) + \delta\right]} \text{ where } \tilde{d}_1 = \tilde{d}_1\left(\Psi_1(\tilde{e}), \Psi_2(\tilde{e}), \tilde{x}_1, \tilde{x}_2\right) \qquad \text{(G.3)} \end{aligned}$$

Hence, to find the unique solution for employment we use Eq. (G.2) and Eq. (G.3) into Eq. (8), which yields

$$\begin{aligned} \tilde{e} &= 2\tilde{d}_2 + \tilde{d}_1 \\ \tilde{e} &= 2\tilde{d}_2\left(\Psi_1(\tilde{e}), \Psi_2(\tilde{e}), \tilde{x}_1, \tilde{x}_2\right) + \tilde{d}_1\left(\Psi_1(\tilde{e}), \Psi_2(\tilde{e}), \tilde{x}_1, \tilde{x}_2\right) \qquad \text{(G.4)} \end{aligned}$$

On the job search decision side, recall Eq. (23) and Eq. (24), where $0 < \tilde{x}_1(\tilde{e}), \tilde{x}_2(\tilde{e}) \geqslant 1/2$.

To prove the existence and uniqueness of the equilibrium using a fixed point argument we must find an expression for employment ($\tilde{e}$ see Eq. G.4) in function of workers' choosiness ($\tilde{x}_1(\Psi_1(\tilde{e}))$ and $\tilde{x}_2(\Psi_2(\tilde{e}))$, see Eqs. (23) and (24)). In other words, G.4 writes:

$$\tilde{e} = 2\tilde{d}_2\left(\Psi_1(\tilde{e}), \Psi_2(\tilde{e}), \tilde{x}_1\left(\Psi_1(\tilde{e})\right), \tilde{x}_2\left(\Psi_2(\tilde{e})\right)\right) + \tilde{d}_1\left(\Psi_1(\tilde{e}), \Psi_2(\tilde{e}), \tilde{x}_1\left(\Psi_1(\tilde{e})\right), \tilde{x}_2\left(\Psi_2(\tilde{e})\right)\right) \qquad \text{(G.5)}$$

For compactness at computing $d\tilde{e}$ we define the global rates of leaving unemployment as $\Psi_1 = \Psi_1(\tilde{e})$ and $\Psi_2 = \Psi_2(\tilde{e})$. Similarly, using compact notation for workers' choosiness as $\tilde{x}_1 = \tilde{x}_1\left(\Psi_1(\tilde{e})\right)$ and $\tilde{x}_2 = \tilde{x}_2\left(\Psi_2(\tilde{e})\right)$, the first derivative of G.5 writes

$$\begin{aligned} d\tilde{e} &= 2\left[\frac{\partial\tilde{d}_2}{\partial\Psi_1}\frac{\partial\Psi_1}{\partial\tilde{e}} + \frac{\partial\tilde{d}_2}{\partial\Psi_2}\frac{\partial\Psi_2}{\partial\tilde{e}} + \frac{\partial\tilde{d}_2}{\partial\tilde{x}_1}\frac{\partial\tilde{x}_1}{\partial\Psi_1}\frac{\partial\Psi_1}{\partial\tilde{e}} + \frac{\partial\tilde{d}_2}{\partial\tilde{x}_2}\frac{\partial\tilde{x}_2}{\partial\Psi_2}\frac{\partial\Psi_2}{\partial\tilde{e}}\right] \\ &\quad + \left[\frac{\partial\tilde{d}_1}{\partial\Psi_1}\frac{\partial\Psi_1}{\partial\tilde{e}} + \frac{\partial\tilde{d}_1}{\partial\Psi_2}\frac{\partial\Psi_2}{\partial\tilde{e}} + \frac{\partial\tilde{d}_1}{\partial\tilde{x}_1}\frac{\partial\tilde{x}_1}{\partial\Psi_1}\frac{\partial\Psi_1}{\partial\tilde{e}} + \frac{\partial\tilde{d}_1}{\partial\tilde{x}_2}\frac{\partial\tilde{x}_2}{\partial\Psi_2}\frac{\partial\Psi_2}{\partial\tilde{e}}\right] \end{aligned}$$

As it is shown below $\frac{\partial\Psi_1}{\partial\tilde{e}} = \frac{\partial\Psi_2}{\partial\tilde{e}}$, thus we factorize this term and obtain

$$
\begin{aligned}
d\tilde{e} &= \frac{\partial\Psi_1}{\partial\tilde{e}}\Bigg\{2\left[\frac{\partial\tilde{d}_2}{\partial\Psi_1} + \frac{\partial\tilde{d}_2}{\partial\Psi_2} + \frac{\partial\tilde{d}_2}{\partial\tilde{x}_1}\frac{\partial\tilde{x}_1}{\partial\Psi_1} + \frac{\partial\tilde{d}_2}{\partial\tilde{x}_2}\frac{\partial\tilde{x}_2}{\partial\Psi_2}\right] \\
&\quad + \left[\frac{\partial\tilde{d}_1}{\partial\Psi_1} + \frac{\partial\tilde{d}_1}{\partial\Psi_2} + \frac{\partial\tilde{d}_1}{\partial\tilde{x}_1}\frac{\partial\tilde{x}_1}{\partial\Psi_1} + \frac{\partial\tilde{d}_1}{\partial\tilde{x}_2}\frac{\partial\tilde{x}_2}{\partial\Psi_2}\right]\Bigg\}
\end{aligned}
$$

As it will be shown below $\frac{\partial\tilde{d}_2}{\partial\Psi_1} = \frac{\partial\tilde{d}_2}{\partial\tilde{x}_1}\frac{\tilde{x}_1}{\Psi_1}$, $\frac{\partial\tilde{d}_2}{\partial\Psi_2} = \frac{\partial\tilde{d}_2}{\partial\tilde{x}_2}\frac{\tilde{x}_2}{\Psi_2}$, $\frac{\partial\tilde{d}_1}{\partial\Psi_1} = \frac{\partial\tilde{d}_1}{\partial\tilde{x}_1}\frac{\tilde{x}_1}{\Psi_1}$, $\frac{\partial\tilde{d}_1}{\partial\Psi_2} = \frac{\partial\tilde{d}_1}{\partial\tilde{x}_2}\frac{\tilde{x}_2}{\Psi_2}$ hence,

$$
\begin{aligned}
d\tilde{e} &= \frac{\partial\Psi_1}{\partial\tilde{e}}\Bigg\{2\left[\frac{\partial\tilde{d}_2}{\partial\tilde{x}_1}\frac{\tilde{x}_1}{\Psi_1} + \frac{\partial\tilde{d}_2}{\partial\tilde{x}_2}\frac{\tilde{x}_2}{\Psi_2} + \frac{\partial\tilde{d}_2}{\partial\tilde{x}_1}\frac{\partial\tilde{x}_1}{\partial\Psi_1} + \frac{\partial\tilde{d}_2}{\partial\tilde{x}_2}\frac{\partial\tilde{x}_2}{\partial\Psi_2}\right] \\
&\quad + \left[\frac{\partial\tilde{d}_1}{\partial\tilde{x}_1}\frac{\tilde{x}_1}{\partial\Psi_1} + \frac{\partial\tilde{d}_1}{\partial\tilde{x}_2}\frac{\tilde{x}_2}{\Psi_2} + \frac{\partial\tilde{d}_1}{\partial\tilde{x}_1}\frac{\partial\tilde{x}_1}{\partial\Psi_1} + \frac{\partial\tilde{d}_1}{\partial\tilde{x}_2}\frac{\partial\tilde{x}_2}{\Psi_2}\right]\Bigg\}
\end{aligned}
$$

Factorizing,

$$
\begin{aligned}
d\tilde{e} &= \frac{\partial\Psi_1}{\partial\tilde{e}}\Bigg\{2\left[\frac{\partial\tilde{d}_2}{\partial\tilde{x}_1}\left(\frac{\tilde{x}_1}{\Psi_1} + \frac{\partial\tilde{x}_1}{\partial\Psi_1}\right) + \frac{\partial\tilde{d}_2}{\partial\tilde{x}_2}\left(\frac{\tilde{x}_2}{\Psi_2} + \frac{\partial\tilde{x}_2}{\partial\Psi_2}\right)\right] \\
&\quad + \left[\frac{\partial\tilde{d}_1}{\partial\tilde{x}_1}\left(\frac{\tilde{x}_1}{\Psi_1} + \frac{\partial\tilde{x}_1}{\partial\Psi_1}\right) + \frac{\partial\tilde{d}_1}{\partial\tilde{x}_2}\left(\frac{\tilde{x}_2}{\Psi_2} + \frac{\partial\tilde{x}_2}{\partial\Psi_2}\right)\right]\Bigg\}
\end{aligned}
$$

Rearranging,

$$
d\tilde{e} = \underbrace{\frac{\partial\Psi_1}{\partial\tilde{e}}}_{+}\left\{\left(2\underbrace{\frac{\partial\tilde{d}_2}{\partial\tilde{x}_1}}_{+} + \underbrace{\frac{\partial\tilde{d}_1}{\partial\tilde{x}_1}}_{+}\right)\left(\underbrace{\frac{\tilde{x}_1}{\Psi_1}}_{+} + \underbrace{\frac{\partial\tilde{x}_1}{\partial\Psi_1}}_{-}\right) + \left(2\underbrace{\frac{\partial\tilde{d}_2}{\partial\tilde{x}_2}}_{+} + \underbrace{\frac{\partial\tilde{d}_1}{\partial\tilde{x}_2}}_{-}\right)\left(\underbrace{\frac{\tilde{x}_2}{\Psi_2}}_{+} + \underbrace{\frac{\partial\tilde{x}_2}{\partial\Psi_2}}_{-}\right)\right\} \tag{G.6}
$$

To determine the sign of $d\tilde{e}$ we study the following terms:

- $\frac{\tilde{x}_1}{\Psi_1} + \frac{\partial\tilde{x}_1}{\partial\Psi_1} = \frac{\tilde{x}_1}{\Psi_1} - \frac{\tau\tilde{x}_1^2}{2\Psi_1\tau\tilde{x}_1+(r+\delta)\tau} = \frac{\tilde{x}_1}{\Psi_1[2\Psi_1\tau\tilde{x}_1+(r+\delta)\tau]}\left[\Psi_1\tau\tilde{x}_1 + (r+\delta)\tau\right] > 0$
- $\frac{\tilde{x}_2}{\Psi_2} + \frac{\partial\tilde{x}_2}{\partial\Psi_2} = \frac{\tilde{x}_2}{\Psi_2} - \frac{\tau\tilde{x}_2^2}{2\Psi_2\tau\tilde{x}_2+(r+\delta)\tau} = \frac{\tilde{x}_2}{\Psi_2[2\Psi_2\tau\tilde{x}_2+(r+\delta)\tau]}\left[\Psi_2\tau\tilde{x}_2 + (r+\delta)\tau\right] > 0$
- $2\frac{\partial\tilde{d}_2}{\partial\tilde{x}_2} + \frac{\partial\tilde{d}_1}{\partial\tilde{x}_2} = 2\frac{2\tilde{x}_1\Psi_1\Psi_2\delta(\delta+4\tilde{x}_1\Psi_1)}{[\delta^2+4\tilde{x}_1\Psi_1[\tilde{x}_2\Psi_2+\delta]]^2} - \frac{8\tilde{x}_1^2\Psi_1^2\Psi_2\delta}{[\delta^2+4\tilde{x}_1\Psi_1[\tilde{x}_2\Psi_2+\delta]]^2} = \frac{4\tilde{x}_1\Psi_1\Psi_2\delta(\delta+4\tilde{x}_1\Psi_1)}{[\delta^2+4\tilde{x}_1\Psi_1[\tilde{x}_2\Psi_2+\delta]]^2} > 0$

Hence, $d\tilde{e} > 0$ and $d^2\tilde{e} = 0$ because $\frac{\partial^2\Psi_1}{\partial\tilde{e}^2} = 0$, which means that the function (G.5) is monotonically increasing.

Substitute Eq. (G.2) and Eq. (G.3) into (G.4) yields

$$
\begin{aligned}
\tilde{e} &= 2\frac{2\tilde{x}_1\tilde{x}_2\Psi_1(\tilde{e})\Psi_2(\tilde{e})}{\delta^2 + 4\tilde{x}_1\Psi_1(\tilde{e})\cdot[\tilde{x}_2\Psi_2(\tilde{e}) + \delta]} + \frac{2\tilde{x}_1\Psi_1(\tilde{e})\delta}{\delta^2 + 4\tilde{x}_1\Psi_1(\tilde{e})\cdot[\tilde{x}_2\Psi_2(\tilde{e}) + \delta]} \\
\tilde{e} &= \frac{4\tilde{x}_1\tilde{x}_2\Psi_1(\tilde{e})\Psi_2(\tilde{e}) + 2\tilde{x}_1\Psi_1(\tilde{e})\delta}{\delta^2 + 4\tilde{x}_1\tilde{x}_2\Psi_1(\tilde{e})\Psi_2(\tilde{e}) + 4\tilde{x}_1\Psi_1(\tilde{e})\delta}
\end{aligned} \tag{G.7}
$$

we determine that RHS of the function (G.7) is lower than one $\forall e$. This is to prove that

$$\begin{aligned}
\frac{4\tilde{x}_1\tilde{x}_2\Psi_1(\tilde{e})\Psi_2(\tilde{e}) + 2\tilde{x}_1\Psi_1(\tilde{e})\delta}{\delta^2 + 4\tilde{x}_1\tilde{x}_2\Psi_1(\tilde{e})\Psi_2(\tilde{e}) + 4\tilde{x}_1\Psi_1(\tilde{e})\delta} &< 1 \\
4\tilde{x}_1\tilde{x}_2\Psi_1(\tilde{e})\Psi_2(\tilde{e}) + 2\tilde{x}_1\Psi_1(\tilde{e})\delta &< \delta^2 + 4\tilde{x}_1\tilde{x}_2\Psi_1(\tilde{e})\Psi_2(\tilde{e}) + 4\tilde{x}_1\Psi_1(\tilde{e})\delta \\
0 &< \delta^2 + 2\tilde{x}_1\Psi_1(\tilde{e})\delta \\
0 &< \delta + 2\tilde{x}_1\Psi_1(\tilde{e}) \\
0 &< \delta + 2\tilde{x}_1\left(\omega e\lambda + \psi\right)
\end{aligned}$$

where $\forall e$ $0 < \tilde{x}_1 \leqslant 1/2$ and by definition $\delta > 0$, $\lambda > 0$ and $\psi > 0$. Hence, the RHS of (G.7) is always lower one $\forall e$.

Next, we determine that RHS of the function (G.7) is positive $\forall e$. This is to prove that

$$\begin{aligned}
0 &< \frac{4\tilde{x}_1\tilde{x}_2\Psi_1(\tilde{e})\Psi_2(\tilde{e}) + 2\tilde{x}_1\Psi_1(\tilde{e})\delta}{\delta^2 + 4\tilde{x}_1\tilde{x}_2\Psi_1(\tilde{e})\Psi_2(\tilde{e}) + 4\tilde{x}_1\Psi_1(\tilde{e})\delta} \\
0 &< 4\tilde{x}_1\tilde{x}_2\Psi_1(\tilde{e})\Psi_2(\tilde{e}) + 2\tilde{x}_1\Psi_1(\tilde{e})\delta \\
0 &< 2\tilde{x}_1\Psi_1(\tilde{e})\left(2\tilde{x}_2\Psi_2(\tilde{e}) + \delta\right) \\
0 &< 2\tilde{x}_2\Psi_2(\tilde{e}) + \delta \\
0 &< 2\tilde{x}_2\left[\left(1 - \omega + \omega e\right)\lambda + \psi\right] + \delta
\end{aligned}$$

where $\forall e$ $0 < \tilde{x}_2 \leqslant 1/2$ and by definition $\delta > 0$, $\lambda > 0$ and $\psi > 0$. Hence, the RHS of (G.7) is strictly positive $\forall e$.

In conclusion the RHS of (G.5) is an increasing function of $\tilde{e}$ between zero and one for $\tilde{e} = 0$ and $\tilde{e} = 1$. Therefore, using a fixed point argument, we have proven that there exists a unique equilibrium.

G.1. Detailed computations of the derivatives

Substituting Eq. (G.1) into Eq. (4) yields

$$\tilde{d}_2 = \frac{2\tilde{x}_1\tilde{x}_2\Psi_1\Psi_2}{\delta^2 + 4\tilde{x}_1\Psi_1\left[\tilde{x}_2\Psi_2 + \delta\right]} \tag{G.8}$$

Evaluating Eq. (G.8)

- $\frac{\partial\tilde{d}_2}{\partial\tilde{x}_1} = \frac{2\tilde{x}_2\Psi_1\Psi_2\delta^2}{\left[\delta^2 + 4\tilde{x}_1\Psi_1[\tilde{x}_2\Psi_2+\delta]\right]^2} > 0$
- $\frac{\partial\tilde{d}_2}{\partial\tilde{x}_2} = \frac{2\tilde{x}_1\Psi_1\Psi_2\delta(\delta+4\tilde{x}_1\Psi_1)}{\left[\delta^2 + 4\tilde{x}_1\Psi_1[\tilde{x}_2\Psi_2+\delta]\right]^2} > 0$

- $\frac{\partial \tilde{d}_2}{\partial \Psi_1} = \frac{2\tilde{x}_1\tilde{x}_2\Psi_2\delta^2}{[\delta^2+4\tilde{x}_1\Psi_1[\tilde{x}_2\Psi_2+\delta]]^2} = \frac{\partial \tilde{d}_2}{\partial \tilde{x}_1}\frac{\tilde{x}_1}{\Psi_1} > 0$
- $\frac{\partial \tilde{d}_2}{\partial \Psi_2} = \frac{2\tilde{x}_1\tilde{x}_2\Psi_1\delta(\delta+4\tilde{x}_1\Psi_1)}{[\delta^2+4\tilde{x}_1\Psi_1[\tilde{x}_2\Psi_2+\delta]]^2} = \frac{\partial \tilde{d}_2}{\partial \tilde{x}_2}\frac{\tilde{x}_2}{\Psi_2} > 0$

By the definition of $\Psi_1(\tilde{e}) = \omega e\lambda + \psi$ and $\Psi_2(\tilde{e}) = (1-\omega+\omega e)\,\lambda + \psi$

- $\frac{\partial \Psi_1}{\partial \tilde{e}} = \frac{\partial \Psi_2}{\partial \tilde{e}} = \omega\lambda > 0$

Hence,

- $\frac{\partial \tilde{d}_2}{\partial \tilde{e}} = \frac{\partial \tilde{d}_2}{\partial \Psi_1}\frac{\partial \Psi_1}{\partial \tilde{e}} + \frac{\partial \tilde{d}_2}{\partial \Psi_2}\frac{\partial \Psi_2}{\partial \tilde{e}} > 0$

Substituting Eq. (G.1) into Eq. (5) yields

$$\tilde{d}_1 = \frac{2\tilde{x}_1\Psi_1\delta}{\delta^2 + 4\tilde{x}_1\Psi_1\left[\tilde{x}_2\Psi_2 + \delta\right]} \tag{G.9}$$

Evaluating Eq. (G.9)

- $\frac{\partial \tilde{d}_1}{\partial \tilde{x}_1} = \frac{2\Psi_1\delta^3}{[\delta^2+4\tilde{x}_1\Psi_1[\tilde{x}_2\Psi_2+\delta]]^2} > 0$
- $\frac{\partial \tilde{d}_1}{\partial \tilde{x}_2} = -\frac{8\tilde{x}_1^2\Psi_1^2\Psi_2\delta}{[\delta^2+4\tilde{x}_1\Psi_1[\tilde{x}_2\Psi_2+\delta]]^2} < 0$
- $\frac{\partial \tilde{d}_1}{\partial \Psi_1} = \frac{2\tilde{x}_1\delta^3}{[\delta^2+4\tilde{x}_1\Psi_1[\tilde{x}_2\Psi_2+\delta]]^2} = \frac{\partial \tilde{d}_1}{\partial \tilde{x}_1}\frac{\tilde{x}_1}{\partial \Psi_1} > 0$
- $\frac{\partial \tilde{d}_1}{\partial \Psi_2} = -\frac{8\tilde{x}_1^2\tilde{x}_2\Psi_1^2\delta}{[\delta^2+4\tilde{x}_1\Psi_1[\tilde{x}_2\Psi_2+\delta]]^2} = \frac{\partial \tilde{d}_1}{\partial \tilde{x}_2}\frac{\tilde{x}_2}{\partial \Psi_2} < 0$
- $\frac{\partial \tilde{d}_1}{\partial \tilde{e}} = \frac{\partial \tilde{d}_1}{\partial \Psi_1}\frac{\partial \Psi_1}{\partial \tilde{e}} + \frac{\partial \tilde{d}_1}{\partial \Psi_2}\frac{\partial \Psi_2}{\partial \tilde{e}} \gtreqless 0$

Evaluating Eq. (23) and Eq. (24) using the implicit function theorem yields

- $\frac{\partial \tilde{x}_2}{\partial \tilde{e}} = -\frac{\partial f_2/\partial \tilde{e}}{\partial f_2/\partial \tilde{x}_2} = -\frac{\partial \Psi_2}{\partial \tilde{e}}\frac{\tau\tilde{x}_2^2}{2\Psi_2\tau\tilde{x}_2+(r+\delta)\tau} < 0$

- $\frac{\partial \tilde{x}_1}{\partial \tilde{e}} = -\frac{\partial f_1/\partial \tilde{e}}{\partial f_1/\partial \tilde{x}_1} = -\frac{\partial \Psi_1}{\partial \tilde{e}}\frac{\tau\tilde{x}_1^2}{2\Psi_1\tau\tilde{x}_1+(r+\delta)\tau} < 0$

H. Totally forward looking dyads: Equilibrium $\tilde{x}_2$

For simplicity we write $\Psi_2(e) = \Psi_2$. First, we evaluate Eq. (27) at $\tilde{x}_2$ and establish the equilibrium condition $W_{11}(\tilde{x}_2) - W_{01} = 0 \Rightarrow \tilde{x}_2$:

$$\begin{aligned} (r+2\delta)\left[W_{11}(\tilde{x}_2) - W_{01}\right] &= w - \tau\cdot\tilde{x}_2 + \delta W_{10}(\tilde{x}_2) - (r+\delta)W_{01} \\ (r+\delta)W_{01} &= w - \tau\cdot\tilde{x}_2 + \delta W_{10}(\tilde{x}_2) \end{aligned} \tag{H.1}$$

Isolating W_{01} from Eq. (28)

$$rW_{01} = b + \Psi_2 \int_{-\tilde{x}_2}^{\tilde{x}_2} \left[W_{11}(x) - W_{01}\right]\partial x - \delta\left[W_{01} - W_{00}\right] \tag{H.2}$$

Substituting Eq. (E.3) into Eq. (H.2) we obtain

$$(r + \delta + 2\tilde{x}_2\Psi_2)\,W_{01} = b + 2\tilde{x}_2\Psi_2 W_{11}(\tilde{x}_2/2) - \delta W_{00} \tag{H.3}$$

Hence, we must find an expression for $W_{11}(\tilde{x}_2/2)$. For this purpose we solve the following system for $W_{11}(\tilde{x}_2/2)$:

$$(r+2\delta)\,W_{11}(\tilde{x}_2/2) = w - \tau\cdot\tilde{x}_2/2 + \delta\left[W_{01} + W_{10}(\tilde{x}_2/2)\right] \tag{H.4}$$

$$(r + \delta + 2\tilde{x}_2\Psi_2)\,W_{10}(\tilde{x}_2/2) = w - \tau\cdot\tilde{x}_2/2 + \delta W_{00} + 2\tilde{x}_2\Psi_2 W_{11}(\tilde{x}_2/2) \tag{H.5}$$

Plugging Eq. (H.5) into (H.4) we obtain

$$\begin{aligned}
(r+2\delta)\,W_{11}\left(\tilde{x}_2/2\right) &= w-\tau\cdot\tilde{x}_2/2+\delta\left[W_{01}+\frac{w-\tau\cdot\tilde{x}_2/2+\delta W_{00}+2\tilde{x}_2\Psi_2 W_{11}\left(\tilde{x}_2/2\right)}{\left(r+\delta+2\tilde{x}_2\Psi_2\right)}\right]\\
\left[(r+2\delta)-\frac{2\tilde{x}_2\Psi_2\delta}{\left(r+\delta+2\tilde{x}_2\Psi_2\right)}\right]W_{11}\left(\tilde{x}_2/2\right) &= \left(w-\tau\cdot\tilde{x}_2/2\right)\left(1+\frac{\delta}{\left(r+\delta+2\tilde{x}_2\Psi_2\right)}\right)+\delta W_{01}+\frac{\delta^2 W_{00}}{\left(r+\delta+2\tilde{x}_2\Psi_2\right)}\\
\left[\frac{(r+2\delta)\,(r+\delta)+2\tilde{x}_2\Psi_2\,(r+2\delta)-2\tilde{x}_2\Psi_2\delta}{\left(r+\delta+2\tilde{x}_2\Psi_2\right)}\right]W_{11}\left(\tilde{x}_2/2\right) &= \frac{\left(w-\tau\cdot\tilde{x}_2/2\right)\left(r+2\delta+2\tilde{x}_2\Psi_2\right)}{\left(r+\delta+2\tilde{x}_2\Psi_2\right)}+\frac{\delta W_{01}\left(r+\delta+2\tilde{x}_2\Psi_2\right)+\delta^2 W_{00}}{\left(r+\delta+2\tilde{x}_2\Psi_2 f\right)}\\
\left[\frac{(r+\delta)\left(r+2\delta+2\tilde{x}_2\Psi_2\right)}{\left(r+\delta+2\tilde{x}_2\Psi_2\right)}\right]W_{11}\left(\tilde{x}_2/2\right) &= \frac{\left(w-\tau\cdot\tilde{x}_2/2\right)\left(r+2\delta+2\tilde{x}_2\Psi_2\right)}{\left(r+\delta+2\tilde{x}_2\Psi_2\right)}+\frac{\delta W_{01}\left(r+\delta+2\tilde{x}_2\Psi_2\right)+\delta^2 W_{00}}{\left(r+\delta+2\tilde{x}_2\Psi_2\right)}\\
(r+\delta)\left(r+2\delta+2\tilde{x}_2\Psi_2\right)W_{11}\left(\tilde{x}_2/2\right) &= \left(w-\tau\cdot\tilde{x}_2/2\right)\left(r+2\delta+2\tilde{x}_2\Psi_2\right)+\delta W_{01}\left(r+\delta+2\tilde{x}_2\Psi_2\right)+\delta^2 W_{00}\\
W_{11}\left(\tilde{x}_2/2\right) &= \frac{\left(w-\tau\cdot\tilde{x}_2/2\right)\left(r+2\delta+2\tilde{x}_2\Psi_2\right)+\delta W_{01}\left(r+\delta+2\tilde{x}_2\Psi_2\right)+\delta^2 W_{00}}{(r+\delta)\left(r+2\delta+2\tilde{x}_2\Psi_2\right)} \qquad \text{(H.6)}
\end{aligned}$$

Substituting (H.6) into (H.3) yields

$$\left(r+\delta+2\tilde{x}_2\Psi_2\right)W_{01}=b+2\tilde{x}_2\Psi_2\left[\frac{\left(w-\tau\cdot\tilde{x}_2/2\right)\left(r+2\delta+2\tilde{x}_2\Psi_2\right)+\delta W_{01}\left(r+\delta+2\tilde{x}_2\Psi_2\right)+\delta^2 W_{00}}{(r+\delta)\left(r+2\delta+2\tilde{x}_2\Psi_2\right)}\right]-\delta W_{00}$$

Isolating W_{01} and factorizing W_{00} yields

$$\begin{aligned}
\left[(r+\delta+2\tilde{x}_2\Psi_2)-\frac{2\tilde{x}_2\Psi_2\delta\,(r+\delta+2\tilde{x}_2\Psi_2)}{(r+\delta)\,(r+2\delta+2\tilde{x}_2\Psi_2)}\right]W_{01} &= b+\frac{2\tilde{x}_2\Psi_2\,(w-\tau\cdot\tilde{x}_2/2)\,(r+2\delta+2\tilde{x}_2\Psi_2)}{(r+\delta)\,(r+2\delta+2\tilde{x}_2\Psi_2)} \\
&\quad -\delta W_{00}\left[1-\frac{2\tilde{x}_2\Psi_2\delta}{(r+\delta)\,(r+2\delta+2\tilde{x}_2\Psi_2)}\right] \\
(r+\delta+2\tilde{x}_2\Psi_2)\left[\frac{(r+\delta)\,(r+2\delta+2\tilde{x}_2\Psi_2)-2\tilde{x}_2\Psi_2\delta}{(r+\delta)\,(r+2\delta+2\tilde{x}_2\Psi_2)}\right]W_{01} &= b+\frac{2\tilde{x}_2\Psi_2\,(w-\tau\cdot\tilde{x}_2/2)\,(r+2\delta+2\tilde{x}_2\Psi_2)}{(r+\delta)\,(r+2\delta+2\tilde{x}_2\Psi_2)} \\
&\quad -\delta W_{00}\left[\frac{(r+\delta)\,(r+2\delta+2\tilde{x}_2\Psi_2)-2\tilde{x}_2\Psi_2\delta}{(r+\delta)\,(r+2\delta+2\tilde{x}_2\Psi_2)}\right] \\
(r+\delta+2\tilde{x}_2\Psi_2)\left[(r+\delta)\,(r+2\delta+2\tilde{x}_2\Psi_2)-2\tilde{x}_2\Psi_2\delta\right]W_{01} &= b\,(r+\delta)\,(r+2\delta+2\tilde{x}_2\Psi_2)+2\tilde{x}_2\Psi_2\,(w-\tau\cdot\tilde{x}_2/2)\,(r+2\delta+2\tilde{x}_2\Psi_2) \\
&\quad -\delta W_{00}\left[(r+\delta)\,(r+2\delta+2\tilde{x}_2\Psi_2)-2\tilde{x}_2\Psi_2\delta\right] \\
W_{01} &= \frac{b\,(r+\delta)\,(r+2\delta+2\tilde{x}_2\Psi_2)+2\tilde{x}_2\Psi_2\,(w-\tau\cdot\tilde{x}_2/2)\,(r+2\delta+2\tilde{x}_2\Psi_2)}{(r+\delta+2\tilde{x}_2\Psi_2)\left[(r+\delta)\,(r+2\delta+2\tilde{x}_2\Psi_2)-2\tilde{x}_2\Psi_2\delta\right]}-\frac{\delta W_{00}}{(r+\delta+2\tilde{x}_2\Psi_2)}
\end{aligned}$$

Let denote $D_{01}=(r+\delta+2\tilde{x}_2\Psi_2)\left[(r+\delta)\,(r+2\delta+2\tilde{x}_2\Psi_2)-2\tilde{x}_2\Psi_2\delta\right]$, hence W_{01} writes

$$W_{01}=\frac{b\,(r+\delta)\,(r+2\delta+2\tilde{x}_2\Psi_2)+2\tilde{x}_2\Psi_2\,(w-\tau\cdot\tilde{x}_2/2)\,(r+2\delta+2\tilde{x}_2\Psi_2)}{D_{01}}-\frac{\delta W_{00}}{(r+\delta+2\tilde{x}_2\Psi_2)} \tag{H.7}$$

Since we have W_{01} in function of W_{00}, we need an expression for W_{00}. For this purpose, we use the equilibrium condition for $\tilde{x}_1$, $W_{10}\,(\tilde{x}_1)-W_{00}=0$. Using (E.3) in Eq. (29)

$$\begin{aligned}
(r+\delta)\,W_{10}\,(\tilde{x}_1) &= w-\tau\cdot\tilde{x}_1+\delta W_{00}+2\tilde{x}_2\Psi_2\left[W_{11}\,(\tilde{x}_2/2)-W_{10}\,(\tilde{x}_2/2)\right]+rW_{00}-rW_{00} \\
(r+\delta)\left[W_{10}\,(\tilde{x}_1)-W_{00}\right] &= w-\tau\cdot\tilde{x}_1+2\tilde{x}_2\Psi_2\left[W_{11}\,(\tilde{x}_2/2)-W_{10}\,(\tilde{x}_2/2)\right]-rW_{00} \qquad \text{(H.8)}
\end{aligned}$$

By the equilibrium condition, $W_{10}\,(\tilde{x}_1)-W_{00}=0$, Eq. (H.8) becomes

$$rW_{00}=w-\tau\cdot\tilde{x}_1+2\tilde{x}_2\Psi_2\left[W_{11}\,(\tilde{x}_2/2)-W_{10}\,(\tilde{x}_2/2)\right] \tag{H.9}$$

Let's solve the difference $[W_{11}(\tilde{x}_2/2) - W_{10}(\tilde{x}_2/2)]$ by using Eqs. (27) and (29) evaluated at $\tilde{x}_2$

$$\begin{aligned} (r+2\delta) W_{11}(\tilde{x}_2/2) &= w - \tau \cdot \tilde{x}_2/2 + \delta\left[W_{01} + W_{10}(\tilde{x}_2/2)\right] & \text{(H.10)} \\ (r+\delta+2\tilde{x}_2\Psi_2) W_{10}(\tilde{x}_2/2) &= w - \tau \cdot \tilde{x}_2/2 + \delta W_{00} + 2\tilde{x}_2\Psi_2 W_{11}(\tilde{x}_2/2) & \text{(H.11)} \end{aligned}$$

Subtracting Eq. (H.11) from Eq. (H.10) yields

$$[W_{11}(\tilde{x}_2/2) - W_{10}(\tilde{x}_2/2)] = \frac{\delta(W_{01} - W_{00})}{(r+2\delta+2\tilde{x}_2\Psi_2)} \tag{H.12}$$

Plugging Eq. (H.12) into Eq. (H.9) we obtain

$$\begin{aligned} rW_{00} &= w - \tau \cdot \tilde{x}_1 + 2\tilde{x}_2\Psi_2 \frac{\delta(W_{01} - W_{00})}{(r+2\delta+2\tilde{x}_2\Psi_2)} \\ \left(r + \frac{2\tilde{x}_2\Psi_2\delta}{(r+2\delta+2\tilde{x}_2\Psi_2)}\right) W_{00} &= \frac{(w-\tau\cdot\tilde{x}_1)(r+2\delta+2\tilde{x}_2\Psi_2) + 2\tilde{x}_2\Psi_2\delta W_{01}}{(r+2\delta+2\tilde{x}_2\Psi_2)} \\ \left(\frac{r(r+2\delta+2\tilde{x}_2\Psi_2) + 2\tilde{x}_2\Psi_2\delta}{(r+2\delta+2\tilde{x}_2\Psi_2)}\right) W_{00} &= \frac{(w-\tau\cdot\tilde{x}_1)(r+2\delta+2\tilde{x}_2\Psi_2) + 2\tilde{x}_2\Psi_2\delta W_{01}}{(r+2\delta+2\tilde{x}_2\Psi_2)} \\ \left[r(r+2\delta+2\tilde{x}_2\Psi_2) + 2\tilde{x}_2\Psi_2\delta\right] W_{00} &= (w-\tau\cdot\tilde{x}_1)(r+2\delta+2\tilde{x}_2\Psi_2) + 2\tilde{x}_2\Psi_2\delta W_{01} \\ W_{00} &= \frac{(w-\tau\cdot\tilde{x}_1)(r+2\delta+2\tilde{x}_2\Psi_2) + 2\tilde{x}_2\Psi_2\delta W_{01}}{\left[r(r+2\delta+2\tilde{x}_2\Psi_2) + 2\tilde{x}_2\Psi_2\delta\right]} \qquad \text{(H.13)} \end{aligned}$$

Let denote $D_{00} = [r(r+2\delta+2\tilde{x}_2\Psi_2) + 2\tilde{x}_2\Psi_2\delta]$, hence Eq. (H.13) writes

$$W_{00} = \frac{(w-\tau\cdot\tilde{x}_1)(r+2\delta+2\tilde{x}_2\Psi_2)}{D_{00}} + \frac{2\tilde{x}_2\Psi_2\delta W_{01}}{D_{00}} \tag{H.14}$$

Substituting Eq. (H.14) into Eq. (H.7) and isolating W_{01} yields

$$\begin{aligned}
W_{01} &= \frac{b\left(r+\delta\right)\left(r+2\delta+2\tilde{x}_2\Psi_2\right)+2\tilde{x}_2\Psi_2\left(w-\tau\cdot\tilde{x}_2/2\right)\left(r+2\delta+2\tilde{x}_2\Psi_2\right)}{D_{01}} \\
&\quad -\frac{\delta}{\left(r+\delta+2\tilde{x}_2\Psi_2\right)}\left[\frac{\left(w-\tau\cdot\tilde{x}_1\right)\left(r+2\delta+2\tilde{x}_2\Psi_2\right)}{D_{00}}+\frac{2\tilde{x}_2\Psi_2\delta W_{01}}{D_{00}}\right] \\
\left[1+\frac{2\tilde{x}_2\Psi_2\delta^2}{D_{00}\left(r+\delta+2\tilde{x}_2\Psi_2\right)}\right]W_{01} &= \frac{b\left(r+\delta\right)\left(r+2\delta+2\tilde{x}_2\Psi_2\right)+2\tilde{x}_2\Psi_2\left(w-\tau\cdot\tilde{x}_2/2\right)\left(r+2\delta+2\tilde{x}_2\Psi_2\right)}{D_{01}} \\
&\quad -\left[\frac{\delta\left(w-\tau\cdot\tilde{x}_1\right)\left(r+2\delta+2\tilde{x}_2\Psi_2\right)}{D_{00}\left(r+\delta+2\tilde{x}_2\Psi_2\right)}\right] \\
\left[\frac{D_{00}\left(r+\delta+2\tilde{x}_2\Psi_2\right)+2\tilde{x}_2\Psi_2\delta^2}{D_{00}\left(r+\delta+2\tilde{x}_2\Psi_2\right)}\right]W_{01} &= \frac{b\left(r+\delta\right)\left(r+2\delta+2\tilde{x}_2\Psi_2\right)+2\tilde{x}_2\Psi_2\left(w-\tau\cdot\tilde{x}_2/2\right)\left(r+2\delta+2\tilde{x}_2\Psi_2\right)}{D_{01}} \\
&\quad -\frac{\delta\left(w-\tau\cdot\tilde{x}_1\right)\left(r+2\delta+2\tilde{x}_2\Psi_2\right)}{D_{00}\left(r+\delta+2\tilde{x}_2\Psi_2\right)}
\end{aligned}$$

where the common denominator on the RHS equals $D_{01}D_{00}\left(r+\delta+2\tilde{x}_2\Psi_2\right)$

$$\begin{aligned}
\left[\frac{D_{00}\left(r+\delta+2\tilde{x}_2\Psi_2\right)+2\tilde{x}_2\Psi_2\delta^2}{D_{00}\left(r+\delta+2\tilde{x}_2\Psi_2\right)}\right]W_{01} &= \frac{\left[b\left(r+\delta\right)\left(r+2\delta+2\tilde{x}_2\Psi_2\right)+2\tilde{x}_2\Psi_2\left(w-\tau\cdot\tilde{x}_2/2\right)\left(r+2\delta+2\tilde{x}_2\Psi_2\right)\right]D_{00}\left(r+\delta+2\tilde{x}_2\Psi_2\right)}{D_{01}D_{00}\left(r+\delta+2\tilde{x}_2\Psi_2\right)} \\
&\quad -\frac{\delta D_{01}\left(w-\tau\cdot\tilde{x}_1\right)\left(r+2\delta+2\tilde{x}_2\Psi_2\right)}{D_{01}D_{00}\left(r+\delta+2\tilde{x}_2\Psi_2\right)}
\end{aligned}$$

Canceling $1/D_{00}\left(r+\delta+2\tilde{x}_2\Psi_2\right)$ from both sides yields

$$\begin{aligned}
D_{01}\left[D_{00}\left(r+\delta+2\tilde{x}_2\Psi_2\right)+2\tilde{x}_2\Psi_2\delta^2\right]W_{01} &= \left(r+2\delta+2\tilde{x}_2\Psi_2\right)\left\{\left[b\left(r+\delta\right)+2\tilde{x}_2\Psi_2\left(w-\tau\cdot\tilde{x}_2/2\right)\right]D_{00}\left(r+\delta+2\tilde{x}_2\Psi_2\right)-\delta D_{01}\left(w-\tau\cdot\tilde{x}_1\right)\right\} \\
W_{01} &= \frac{\left(r+2\delta+2\tilde{x}_2\Psi_2\right)\left\{\left[b\left(r+\delta\right)+2\tilde{x}_2\Psi_2\left(w-\tau\cdot\tilde{x}_2/2\right)\right]D_{00}\left(r+\delta+2\tilde{x}_2\Psi_2\right)-\delta D_{01}\left(w-\tau\cdot\tilde{x}_1\right)\right\}}{D_{01}\left[D_{00}\left(r+\delta+2\tilde{x}_2\Psi_2\right)+2\tilde{x}_2\Psi_2\delta^2\right]}
\end{aligned} \tag{H.15}$$

Let denote the denominator of Eq. (H.15) as $\Omega=D_{01}\left[D_{00}\left(r+\delta+2\tilde{x}_2\Psi_2\right)+2\tilde{x}_2\Psi_2\delta^2\right]$ which in its expanded form writes

$$\Omega=\left(r+\delta+2\tilde{x}_2\Psi_2\right)\left[\left(r+\delta\right)\left(r+2\delta+2\tilde{x}_2\Psi_2\right)-2\tilde{x}_2\Psi_2\delta\right]\left\{\left[r\left(r+2\delta+2\tilde{x}_2\Psi_2\right)+2\tilde{x}_2\Psi_2\delta\right]\left(r+\delta+2\tilde{x}_2\Psi_2\right)+2\tilde{x}_2\Psi_2\delta^2\right\} \tag{H.16}$$

Factorizing (H.16) to build a polynomial expression for $\tilde{x}_2$

$$\begin{aligned}\Omega &= (r+\delta+2\tilde{x}_2\Psi_2)\left[(r+\delta)(r+2\delta+2\tilde{x}_2\Psi_2)-2\tilde{x}_2\Psi_2\delta\right]\left\{\left[r(r+2\delta+2\tilde{x}_2\Psi_2)+2\tilde{x}_2\Psi_2\delta\right](r+\delta+2\tilde{x}_2\Psi_2)+2\tilde{x}_2\Psi_2\delta^2\right\}\\
&= (r+\delta+2\tilde{x}_2\Psi_2)\left[(r+\delta)(r+2\delta)+2\tilde{x}_2\Psi_2(r+\delta)-2\tilde{x}_2\Psi_2\delta\right]\left\{\left[r(r+2\delta)+2\tilde{x}_2\Psi_2 r+2\tilde{x}_2\Psi_2\delta\right](r+\delta+2\tilde{x}_2\Psi_2)+2\tilde{x}_2\Psi_2\delta^2\right\}\\
&= (r+\delta+2\tilde{x}_2\Psi_2)\left[(r+\delta)(r+2\delta)+2\tilde{x}_2\Psi_2 r\right]\left\{\left[r(r+2\delta)+2\tilde{x}_2\Psi_2(r+\delta)\right](r+\delta+2\tilde{x}_2\Psi_2)+2\tilde{x}_2\Psi_2\delta^2\right\}\\
&= (r+\delta+2\tilde{x}_2\Psi_2)\left[(r+\delta)(r+2\delta)+2\tilde{x}_2\Psi_2 r\right]\left\{(r+\delta)\left[r(r+2\delta)+2\tilde{x}_2\Psi_2(r+\delta)\right]+2\tilde{x}_2\Psi_2\left[r(r+2\delta)+2\tilde{x}_2\Psi_2(r+\delta)\right]+2\tilde{x}_2\Psi_2\delta^2\right\}\\
&= (r+\delta+2\tilde{x}_2\Psi_2)\left[(r+\delta)(r+2\delta)+2\tilde{x}_2\Psi_2 r\right]\left\{r(r+\delta)(r+2\delta)+2\tilde{x}_2\Psi_2(r+\delta)^2+2\tilde{x}_2\Psi_2 r(r+2\delta)+(2\tilde{x}_2\Psi_2)^2(r+\delta)+2\tilde{x}_2\Psi_2\delta^2\right\}\\
&= (r+\delta+2\tilde{x}_2\Psi_2)\left[(r+\delta)(r+2\delta)+2\tilde{x}_2\Psi_2 r\right]\left\{(2\tilde{x}_2\Psi_2)^2(r+\delta)+2\tilde{x}_2\Psi_2\left[(r+\delta)^2+r(r+2\delta)+\delta^2\right]+r(r+\delta)(r+2\delta)\right\}\\
&= (r+\delta+2\tilde{x}_2\Psi_2)\left[(r+\delta)(r+2\delta)+2\tilde{x}_2\Psi_2 r\right]\left\{(2\tilde{x}_2\Psi_2)^2(r+\delta)+2\tilde{x}_2\Psi_2\left[r^2+2r\delta+\delta^2+r^2+2r\delta+\delta^2\right]+r(r+\delta)(r+2\delta)\right\}\\
&= (r+\delta+2\tilde{x}_2\Psi_2)\left[(r+\delta)(r+2\delta)+2\tilde{x}_2\Psi_2 r\right]\left\{(2\tilde{x}_2\Psi_2)^2(r+\delta)+2\tilde{x}_2\Psi_2\left[2r^2+4r\delta+2\delta^2\right]+r(r+\delta)(r+2\delta)\right\}\\
&= (r+\delta+2\tilde{x}_2\Psi_2)\left[(r+\delta)(r+2\delta)+2\tilde{x}_2\Psi_2 r\right]\left[(2\tilde{x}_2\Psi_2)^2(r+\delta)+2(2\tilde{x}_2\Psi_2)(r+\delta)^2+r(r+\delta)(r+2\delta)\right]\\
&= (r+\delta)(r+\delta+2\tilde{x}_2\Psi_2)\left[(r+\delta)(r+2\delta)+2\tilde{x}_2\Psi_2 r\right]\left[(2\tilde{x}_2\Psi_2)^2+2(2\tilde{x}_2\Psi_2)(r+\delta)+r(r+2\delta)\right]
\end{aligned}\qquad\text{(H.17)}$$

Let denote $\chi=\left[(r+\delta)(r+2\delta)+2\tilde{x}_2\Psi_2 r\right]\left[(2\tilde{x}_2\Psi_2)^2+2(2\tilde{x}_2\Psi_2)(r+\delta)+r(r+2\delta)\right]$, hence $\Omega=(r+\delta)(r+\delta+2\tilde{x}_2\Psi_2)\cdot\chi$. Factorizing χ to build a polynomial expression for $\tilde{x}_2$ yields

$$\begin{aligned}\chi &= \left[(r+\delta)(r+2\delta)+2\tilde{x}_2\Psi_2 r\right]\left[(2\tilde{x}_2\Psi_2)^2+2(2\tilde{x}_2\Psi_2)(r+\delta)+r(r+2\delta)\right]\\
&= (2\tilde{x}_2\Psi_2)^2(r+\delta)(r+2\delta)+(2\tilde{x}_2\Psi_2)^3 r\\
&\quad+2(2\tilde{x}_2\Psi_2)(r+\delta)^2(r+2\delta)+2(2\tilde{x}_2\Psi_2)^2(r+\delta)r\\
&\quad+r(r+\delta)(r+2\delta)^2+2\tilde{x}_2\Psi_2 r^2(r+2\delta)\\
&= (2\tilde{x}_2\Psi_2)^3 r+(2\tilde{x}_2\Psi_2)^2(r+\delta)\left[2(r+\delta)+r\right]+(2\tilde{x}_2\Psi_2)(r+2\delta)\left[2(r+\delta)^2+r^2\right]+r(r+\delta)(r+2\delta)^2
\end{aligned}$$

Hence, (H.17) becomes

$$\Omega = (r+\delta)(r+\delta+2\tilde{x}_2\Psi_2)\cdot\left\{(2\tilde{x}_2\Psi_2)^3 r + (2\tilde{x}_2\Psi_2)^2 (r+\delta)[2(r+\delta)+r] + (2\tilde{x}_2\Psi_2)(r+2\delta)\left[2(r+\delta)^2+r^2\right] + r(r+\delta)(r+2\delta)^2\right\}$$

and Eq. (H.15) writes

$$W_{01} = \frac{(r+2\delta+2\tilde{x}_2\Psi_2)\{[b(r+\delta)+2\tilde{x}_2\Psi_2(w-\tau\cdot\tilde{x}_2/2)]D_{00}(r+\delta+2\tilde{x}_2\Psi_2)-\delta D_{01}(w-\tau\cdot\tilde{x}_1)\}}{(r+\delta)(r+\delta+2\tilde{x}_2\Psi_2)\cdot\chi} \tag{H.18}$$

We work on the numerator of Eq. (H.18) dropping $(r+2\delta+2\tilde{x}_2\Psi_2)$

$$= [b(r+\delta)+2\tilde{x}_2\Psi_2(w-\tau\cdot\tilde{x}_2/2)]D_{00}(r+\delta+2\tilde{x}_2\Psi_2)-\delta D_{01}(w-\tau\cdot\tilde{x}_1)$$

Recall $D_{00} = [r(r+2\delta)+2\tilde{x}_2\Psi_2(r+\delta)]$ and $D_{01} = \left[(r+\delta)^2(r+2\delta)+2(2\tilde{x}_2\Psi_2)(r+\delta)^2+(2\tilde{x}_2\Psi_2)^2 r\right]$, hence

$$= [b(r+\delta)+2\tilde{x}_2\Psi_2(w-\tau\cdot\tilde{x}_2/2)][r(r+2\delta)+2\tilde{x}_2\Psi_2(r+\delta)](r+\delta+2\tilde{x}_2\Psi_2)$$
$$-\delta\left[(r+\delta)^2(r+2\delta)+2(2\tilde{x}_2\Psi_2)(r+\delta)^2+(2\tilde{x}_2\Psi_2)^2 r\right](w-\tau\cdot\tilde{x}_1)$$

$$= [b(r+\delta)+2\tilde{x}_2\Psi_2 w-2\tilde{x}_2\Psi_2\tau\cdot\tilde{x}_2/2]\left[r(r+\delta)(r+2\delta)+(2\tilde{x}_2\Psi_2)(r+\delta)^2+r(2\tilde{x}_2\Psi_2)(r+2\delta)+(2\tilde{x}_2\Psi_2)^2(r+\delta)\right]$$
$$-\delta(r+\delta)^2(r+2\delta)(w-\tau\cdot\tilde{x}_1)-2\delta(2\tilde{x}_2\Psi_2)(r+\delta)^2(w-\tau\cdot\tilde{x}_1)-(2\tilde{x}_2\Psi_2)^2 r\delta(w-\tau\cdot\tilde{x}_1)$$
$$= \left[b(r+\delta)+(2\tilde{x}_2\Psi_2)w-\tilde{x}_2^2\Psi_2\tau\right]\left[r(r+\delta)(r+2\delta)+(2\tilde{x}_2\Psi_2)\left[(r+\delta)^2+r(r+2\delta)\right]+(2\tilde{x}_2\Psi_2)^2(r+\delta)\right]$$
$$-\delta(r+\delta)^2(r+2\delta)(w-\tau\cdot\tilde{x}_1)-2\delta(2\tilde{x}_2\Psi_2)(r+\delta)^2(w-\tau\cdot\tilde{x}_1)-(2\tilde{x}_2\Psi_2)^2 r\delta(w-\tau\cdot\tilde{x}_1)$$
$$= r(r+\delta)(r+2\delta)b(r+\delta)+(2\tilde{x}_2\Psi_2)b(r+\delta)\left[(r+\delta)^2+r(r+2\delta)\right]+(2\tilde{x}_2\Psi_2)^2(r+\delta)^2 b$$
$$+(2\tilde{x}_2\Psi_2)wr(r+\delta)(r+2\delta)+(2\tilde{x}_2\Psi_2)^2 w\left[(r+\delta)^2+r(r+2\delta)\right]+(2\tilde{x}_2\Psi_2)^3 w(r+\delta)$$
$$-\tilde{x}_2^2\Psi_2\tau r(r+\delta)(r+2\delta)-\tilde{x}_2^3 2\Psi_2^2\tau\left[(r+\delta)^2+r(r+2\delta)\right]-\tilde{x}_2^4 4\tau\Psi_2^3(r+\delta)$$
$$-\delta(r+\delta)^2(r+2\delta)(w-\tau\tilde{x}_1)-4\tilde{x}_2\Psi_2\delta(r+\delta)^2(w-\tau\tilde{x}_1)-\tilde{x}_2^2 4\Psi_2^2 r\delta(w-\tau\tilde{x}_1)$$

Grouping $\tilde{x}_2^4, \tilde{x}_2^3, \tilde{x}_2^2, \tilde{x}_2$

$$\begin{aligned}
=& -\tilde{x}_2^4 4\tau\Psi_2^3 (r+\delta) \\
&+ (2\tilde{x}_2\Psi_2)^3 (r+\delta) w - \tilde{x}_2^3 2\Psi_2^2 \tau \left[(r+\delta)^2 + r(r+2\delta)\right] \\
&+ (2\tilde{x}_2\Psi_2)^2 b (r+\delta)^2 + (2\tilde{x}_2\Psi_2)^2 w \left[(r+\delta)^2 + r(r+2\delta)\right] - \tilde{x}_2^2 \tau\Psi_2 r (r+\delta)(r+2\delta) - (2\tilde{x}_2\Psi_2)^2 r\delta (w - \tau\cdot\tilde{x}_1) \\
&+ (2\tilde{x}_2\Psi_2) b (r+\delta) \left[(r+\delta)^2 + r(r+2\delta)\right] + (2\tilde{x}_2\Psi_2) rw (r+\delta)(r+2\delta) - 4\tilde{x}_2\Psi_2\delta (r+\delta)^2 (w-\tau\cdot\tilde{x}_1) \\
&+ r(r+\delta)^2 (r+2\delta) b - \delta (r+\delta)^2 (r+2\delta)(w-\tau\cdot\tilde{x}_1)
\end{aligned}$$

Factorizing $\tilde{x}_2^4, \tilde{x}_2^3, \tilde{x}_2^2, \tilde{x}_2$

$$\begin{aligned}
=& -\tilde{x}_2^4 4\tau\Psi_2^3 (r+\delta) \\
&+ \tilde{x}_2^3 2\Psi_2^2 \left\{4\Psi_2 w (r+\delta) - \tau\left[(r+\delta)^2 + r(r+2\delta)\right]\right\} \\
&+ \tilde{x}_2^2 \Psi_2 \left\{4\Psi_2 b (r+\delta)^2 + 4\Psi_2 w \left[(r+\delta)^2 + r(r+2\delta)\right] - \tau r (r+\delta)(r+2\delta) - 4\Psi_2 r\delta (w-\tau\cdot\tilde{x}_1)\right\} \\
&+ \tilde{x}_2 2\Psi_2 (r+\delta) \left\{b\left[(r+\delta)^2 + r(r+2\delta)\right] + rw(r+2\delta) - 2\delta(r+\delta)(w-\tau\cdot\tilde{x}_1)\right\} \\
&+ (r+\delta)^2 (r+2\delta)\left[br - \delta(w-\tau\cdot\tilde{x}_1)\right]
\end{aligned}$$

Hence, (H.18) becomes

$$\begin{aligned}
W_{01} =& \{(r+2\delta+2\tilde{x}_2\Psi_2)\{-\tilde{x}_2^4 4\tau\Psi_2^3 (r+\delta) \\
&+ \tilde{x}_2^3 2\Psi_2^2 \left\{4\Psi_2 w (r+\delta) - \tau\left[(r+\delta)^2 + r(r+2\delta)\right]\right\} \\
&+ \tilde{x}_2^2 \Psi_2 \left\{4\Psi_2 b (r+\delta)^2 + 4\Psi_2 w \left[(r+\delta)^2 + r(r+2\delta)\right] - \tau r (r+\delta)(r+2\delta) - 4\Psi_2 r\delta (w-\tau\cdot\tilde{x}_1)\right\} \\
&+ \tilde{x}_2 2\Psi_2 (r+\delta) \left\{b\left[(r+\delta)^2 + r(r+2\delta)\right] + rw(r+2\delta) - 2\delta(r+\delta)(w-\tau\cdot\tilde{x}_1)\right\} \\
&+ (r+\delta)^2 (r+2\delta)\left[br - \delta(w-\tau\cdot\tilde{x}_1)\right]\}\}/(r+\delta)(r+\delta+2\tilde{x}_2\Psi_2)\cdot\chi
\end{aligned} \tag{H.19}$$

b) Expression for $W_{10}(\tilde{x}_2)$ Recall

$$(r+\delta) W_{10}(\tilde{x}_2) = w - \tau \cdot \tilde{x}_2 + \delta W_{00} + 2\tilde{x}_2 \Psi_2 \left[W_{11}(\tilde{x}_2/2) - W_{10}(\tilde{x}_2/2)\right] \tag{H.20}$$

Substituting (H.12) into (H.20)

$$(r+\delta) W_{10}(\tilde{x}_2) = w - \tau \cdot \tilde{x}_2 + \delta W_{00} + \frac{2\tilde{x}_2 \Psi_2 \delta (W_{01} - W_{00})}{(r + 2\delta + 2\tilde{x}_2 \Psi_2)}$$

Factorizing W_{00}

$$(r+\delta) W_{10}(\tilde{x}_2) = w - \tau \cdot \tilde{x}_2 + \delta W_{00} \left[1 - \frac{2\tilde{x}_2 \Psi_2 \delta}{(r + 2\delta + 2\tilde{x}_2 \Psi_2)}\right] + \frac{2\tilde{x}_2 \Psi_2 \delta W_{01}}{(r + 2\delta + 2\tilde{x}_2 \Psi_2)}$$

$$W_{10}(\tilde{x}_2) = \frac{(w - \tau \cdot \tilde{x}_2)(r + 2\delta + 2\tilde{x}_2 \Psi_2) + (r + 2\delta)\delta W_{00} + 2\tilde{x}_2 \Psi_2 \delta W_{01}}{(r+\delta)(r + 2\delta + 2\tilde{x}_2 \Psi_2)} \tag{H.21}$$

Substituting (H.14) into (H.21)

$$W_{10}(\tilde{x}_2) = \frac{(w - \tau \cdot \tilde{x}_2)(r + 2\delta + 2\tilde{x}_2 \Psi_2)}{(r+\delta)(r + 2\delta + 2\tilde{x}_2 \Psi_2)} + \frac{(r + 2\delta)\delta W_{00}}{(r+\delta)(r + 2\delta + 2\tilde{x}_2 \Psi_2)} + \frac{2\tilde{x}_2 \Psi_2 \delta W_{01}}{(r+\delta)(r + 2\delta + 2\tilde{x}_2 \Psi_2)}$$

$$W_{10}(\tilde{x}_2) = \frac{(w - \tau \cdot \tilde{x}_2)}{(r+\delta)} + \frac{(r + 2\delta)\delta}{(r+\delta)(r + 2\delta + 2\tilde{x}_2 \Psi_2)} \left[\frac{(w - \tau \cdot \tilde{x}_1)(r + 2\delta + 2\tilde{x}_2 \Psi_2)}{D_{00}} + \frac{2\tilde{x}_2 \Psi_2 \delta W_{01}}{D_{00}}\right] + \frac{2\tilde{x}_2 \Psi_2 \delta W_{01}}{(r+\delta)(r + 2\delta + 2\tilde{x}_2 \Psi_2)}$$

Factorizing W_{01}

$$W_{10}(\tilde{x}_2) = \frac{(w - \tau \cdot \tilde{x}_2)}{(r+\delta)} + \frac{\delta (w - \tau \cdot \tilde{x}_1)(r + 2\delta)}{D_{00}(r+\delta)} + \frac{2\tilde{x}_2 \Psi_2 \delta^2 (r + 2\delta) W_{01}}{D_{00}(r+\delta)(r + 2\delta + 2\tilde{x}_2 \Psi_2)} + \frac{2\tilde{x}_2 \Psi_2 \delta W_{01}}{(r+\delta)(r + 2\delta + 2\tilde{x}_2 \Psi_2)}$$

$$W_{10}(\tilde{x}_2) = \frac{(w - \tau \cdot \tilde{x}_2)}{(r+\delta)} + \frac{\delta (w - \tau \cdot \tilde{x}_1)(r + 2\delta)}{D_{00}(r+\delta)} + \frac{2\tilde{x}_2 \Psi_2 \delta W_{01}}{(r+\delta)(r + 2\delta + 2\tilde{x}_2 \Psi_2)} \left[\frac{\delta (r + 2\delta)}{D_{00}} + 1\right]$$

$$W_{10}(\tilde{x}_2) = \frac{(w - \tau \cdot \tilde{x}_2)}{(r+\delta)} + \frac{\delta (w - \tau \cdot \tilde{x}_1)(r + 2\delta)}{D_{00}(r+\delta)} + \frac{2\tilde{x}_2 \Psi_2 \delta W_{01} \left[\delta (r + 2\delta) + D_{00}\right]}{D_{00}(r+\delta)(r + 2\delta + 2\tilde{x}_2 \Psi_2)} \tag{H.22}$$

Let's work on $[\delta (r+2\delta) + D_{00}]$

$$\begin{aligned}
[\delta (r+2\delta) + D_{00}] &= \delta (r+2\delta) + r (r+2\delta+2\tilde{x}_2\Psi_2) + 2\tilde{x}_2\Psi_2\delta \\
&= \delta (r+2\delta) + r (r+2\delta) + r2\tilde{x}_2\Psi_2 + 2\tilde{x}_2\Psi_2\delta \\
&= (r+2\delta)(r+\delta) + 2\tilde{x}_2\Psi_2 (r+\delta) \\
&= (r+\delta)(r+2\delta+2\tilde{x}_2\Psi_2)
\end{aligned}$$

Hence (H.22) writes

$$W_{10}(\tilde{x}_2) = \frac{(w-\tau\cdot\tilde{x}_2)}{(r+\delta)} + \frac{\delta (w-\tau\cdot\tilde{x}_1)(r+2\delta)}{D_{00}(r+\delta)} + \frac{2\tilde{x}_2\Psi_2\delta W_{01}[(r+\delta)(r+2\delta+2\tilde{x}_2\Psi_2)]}{D_{00}(r+\delta)(r+2\delta+2\tilde{x}_2\Psi_2)} \tag{H.23}$$

Simplifying the third term of (H.23) yields

$$W_{10}(\tilde{x}_2) = \frac{(w-\tau\cdot\tilde{x}_2)}{(r+\delta)} + \frac{\delta (w-\tau\cdot\tilde{x}_1)(r+2\delta)}{D_{00}(r+\delta)} + \frac{2\tilde{x}_2\Psi_2\delta W_{01}}{D_{00}} \tag{H.24}$$

Hence, Eq. (H.24) is the equation to be substituted in condition (H.1).

c) Plugging Eq. (H.24) into the condition $\tilde{x}_2$, (H.1) yields

$$\begin{aligned}
(r+\delta) W_{01} &= w-\tau\cdot\tilde{x}_2 + \delta W_{10}(\tilde{x}_2) \\
(r+\delta) W_{01} &= w-\tau\cdot\tilde{x}_2 + \delta\left[\frac{(w-\tau\cdot\tilde{x}_2)}{(r+\delta)} + \frac{\delta (w-\tau\cdot\tilde{x}_1)(r+2\delta)}{D_{00}(r+\delta)} + \frac{2\tilde{x}_2\Psi_2\delta W_{01}}{D_{00}}\right]
\end{aligned}$$

Isolating W_{01} we obtain

$$\begin{aligned}
\left[(r+\delta)-\frac{2\tilde{x}_2\Psi_2\delta^2}{D_{00}}\right]W_{01} &= (w-\tau\cdot\tilde{x}_2)\left(1+\frac{\delta}{(r+\delta)}\right)+\frac{\delta^2\,(w-\tau\cdot\tilde{x}_1)\,(r+2\delta)}{D_{00}\,(r+\delta)} \\
\left[\frac{D_{00}\,(r+\delta)-2\tilde{x}_2\Psi_2\delta^2}{D_{00}}\right]W_{01} &= \frac{(w-\tau\cdot\tilde{x}_2)\,(r+2\delta)}{(r+\delta)}+\frac{\delta^2\,(w-\tau\cdot\tilde{x}_1)\,(r+2\delta)}{D_{00}\,(r+\delta)} \\
\left[\frac{D_{00}\,(r+\delta)-2\tilde{x}_2\Psi_2\delta^2}{D_{00}}\right]W_{01} &= \frac{(r+2\delta)}{(r+\delta)}\left[(w-\tau\cdot\tilde{x}_2)+\frac{\delta^2\,(w-\tau\cdot\tilde{x}_1)}{D_{00}}\right] \\
\left[\frac{D_{00}\,(r+\delta)-2\tilde{x}_2\Psi_2\delta^2}{D_{00}}\right]W_{01} &= \frac{(r+2\delta)}{(r+\delta)}\left[\frac{D_{00}\,(w-\tau\cdot\tilde{x}_2)+\delta^2\,(w-\tau\cdot\tilde{x}_1)}{D_{00}}\right] \\
\left[D_{00}\,(r+\delta)-2\tilde{x}_2\Psi_2\delta^2\right]W_{01} &= \frac{(r+2\delta)}{(r+\delta)}\left[D_{00}\,(w-\tau\cdot\tilde{x}_2)+\delta^2\,(w-\tau\cdot\tilde{x}_1)\right] \\
(r+\delta)\left[D_{00}\,(r+\delta)-2\tilde{x}_2\Psi_2\delta^2\right]W_{01} &= (r+2\delta)\left[D_{00}\,(w-\tau\cdot\tilde{x}_2)+\delta^2\,(w-\tau\cdot\tilde{x}_1)\right] \qquad \text{(H.25)}
\end{aligned}$$

Let's study $[D_{00}(r+\delta) - 2\tilde{x}_2\Psi_2\delta^2]$

$$\begin{aligned}
\left[D_{00}(r+\delta) - 2\tilde{x}_2\Psi_2\delta^2\right] &= (r+\delta)\left[r(r+2\delta+2\tilde{x}_2\Psi_2) + 2\tilde{x}_2\Psi_2\delta\right] - 2\tilde{x}_2\Psi_2\delta^2 \\
&= r(r+\delta)(r+2\delta+2\tilde{x}_2\Psi_2) + 2\tilde{x}_2\Psi_2\delta(r+\delta) - 2\tilde{x}_2\Psi_2\delta^2 \\
&= r(r+\delta)(r+2\delta+2\tilde{x}_2\Psi_2) + 2\tilde{x}_2\Psi_2\delta r + 2\tilde{x}_2\Psi_2\delta^2 - 2\tilde{x}_2\Psi_2\delta^2 \\
&= r(r+\delta)(r+2\delta+2\tilde{x}_2\Psi_2) + 2\tilde{x}_2\Psi_2\delta r \\
&= r\left[(r+\delta)(r+2\delta+2\tilde{x}_2\Psi_2) + 2\tilde{x}_2\Psi_2\delta\right] \\
&= r\left[(r+2\delta)(r+\delta) + 2\tilde{x}_2\Psi_2(r+\delta) + 2\tilde{x}_2\Psi_2\delta\right] \\
&= r\left[(r+2\delta)(r+\delta) + 2\tilde{x}_2\Psi_2 r + 2\tilde{x}_2\Psi_2\delta + 2\tilde{x}_2\Psi_2\delta\right] \\
&= r\left[(r+2\delta)(r+\delta) + 2\tilde{x}_2\Psi_2 r + 4\tilde{x}_2\Psi_2\delta\right] \\
&= r\left[(r+2\delta)(r+\delta) + 2\tilde{x}_2\Psi_2(r+2\delta)\right] \\
&= r(r+2\delta)(r+\delta+2\tilde{x}_2\Psi_2)
\end{aligned}$$

Hence (H.25) becomes

$$r(r+\delta)(r+2\delta)(r+\delta+2\tilde{x}_2\Psi_2)W_{01} = (r+2\delta)\left[D_{00}(w-\tau\cdot\tilde{x}_2) + \delta^2(w-\tau\cdot\tilde{x}_1)\right]$$

Simplifying $(r+2\delta)$ we obtain

$$W_{01} = \frac{[D_{00}(w-\tau\cdot\tilde{x}_2) + \delta^2(w-\tau\cdot\tilde{x}_1)]}{r(r+\delta)(r+\delta+2\tilde{x}_2\Psi_2)}$$

$$W_{01} = \frac{[(D_{00}+\delta^2)w - \tau\cdot\tilde{x}_2 D_{00} - \tau\cdot\tilde{x}_1\delta^2]}{r(r+\delta)(r+\delta+2\tilde{x}_2\Psi_2)} \tag{H.26}$$

Recall that $D_{00} = [r(r+2\delta+2\tilde{x}_2\Psi_2)+2\tilde{x}_2\Psi_2\delta] = [r(r+2\delta)+2\tilde{x}_2\Psi_2(r+\delta)]$, hence

$$\begin{aligned}\left[\left(D_{00}+\delta^2\right)w-\tau\cdot\tilde{x}_2D_{00}-\tau\cdot\tilde{x}_1\delta^2\right] &= \left[r(r+2\delta)+2\tilde{x}_2\Psi_2(r+\delta)+\delta^2\right]w-\tau\cdot\tilde{x}_2\left[r(r+2\delta)+2\tilde{x}_2\Psi_2(r+\delta)\right]-\tau\cdot\tilde{x}_1\delta^2\\ &= r(r+2\delta)w+2\tilde{x}_2\Psi_2w(r+\delta)+\delta^2w-\tilde{x}_2\tau r(r+2\delta)-\tilde{x}_2^22\tau\Psi_2(r+\delta)-\tau\tilde{x}_1\delta^2\\ &= -\tilde{x}_2^22\tau\Psi_2(r+\delta)+\tilde{x}_2\left[2\Psi_2w(r+\delta)-\tau r(r+2\delta)\right]+\left[r(r+2\delta)w+\delta^2(w-\tau\tilde{x}_1)\right]\end{aligned}$$

Therefore, Eq. (H.26) becomes

$$W_{01} = \frac{-\tilde{x}_2^22\tau\Psi_2(r+\delta)+\tilde{x}_2\left[2\Psi_2w(r+\delta)-\tau r(r+2\delta)\right]+\left[r(r+2\delta)w+\delta^2(w-\tau\tilde{x}_1)\right]}{r(r+\delta)(r+\delta+2\tilde{x}_2\Psi_2)} \tag{H.27}$$

d) Solution, combining (H.19)

$$\begin{aligned}W_{01} =&\{(r+2\delta+2\tilde{x}_2\Psi_2)\{-\tilde{x}_2^44\tau\Psi_2^3(r+\delta)\\ &+\tilde{x}_2^32\Psi_2^2\left\{4\Psi_2w(r+\delta)-\tau\left[(r+\delta)^2+r(r+2\delta)\right]\right\}\\ &+\tilde{x}_2^2\Psi_2\left\{4\Psi_2b(r+\delta)^2+4\Psi_2w\left[(r+\delta)^2+r(r+2\delta)\right]-\tau r(r+\delta)(r+2\delta)-4\Psi_2r\delta(w-\tau\cdot\tilde{x}_1)\right\}\\ &+\tilde{x}_22\Psi_2(r+\delta)\left\{b\left[(r+\delta)^2+r(r+2\delta)\right]+rw(r+2\delta)-2\delta(r+\delta)(w-\tau\cdot\tilde{x}_1)\right\}\\ &+(r+\delta)^2(r+2\delta)\left[br-\delta(w-\tau\cdot\tilde{x}_1)\right]\}\}/(r+\delta)(r+\delta+2\tilde{x}_2\Psi_2)\cdot\chi\end{aligned}$$

and (H.27)

$$W_{01} = \frac{-\tilde{x}_2^22\tau\Psi_2(r+\delta)+\tilde{x}_2\left[2\Psi_2w(r+\delta)-\tau r(r+2\delta)\right]+\left[r(r+2\delta)w+\delta^2(w-\tau\tilde{x}_1)\right]}{r(r+\delta)(r+\delta+2\tilde{x}_2\Psi_2)}$$

mplifying $1/(r+\delta)(r+\delta+2\tilde{x}_2\Psi_2)$ from (H.19) and (H.27) yields

$$\begin{aligned}
& r(r+2\delta+2\tilde{x}_2\Psi_2)\{-\tilde{x}_2^4 4\tau\Psi_2^3(r+\delta) \\
& + \tilde{x}_2^3 2\Psi_2^2\left\{4\Psi_2 w(r+\delta)-\tau\left[(r+\delta)^2+r(r+2\delta)\right]\right\} \\
& + \tilde{x}_2^2\Psi_2\left\{4\Psi_2 b(r+\delta)^2+4\Psi_2 w\left[(r+\delta)^2+r(r+2\delta)\right]-\tau r(r+\delta)(r+2\delta)-4\Psi_2 r\delta(w-\tau\cdot\tilde{x}_1)\right\} \\
& + \tilde{x}_2 2\Psi_2(r+\delta)\left\{b\left[(r+\delta)^2+r(r+2\delta)\right]+rw(r+2\delta)-2\delta(r+\delta)(w-\tau\cdot\tilde{x}_1)\right\} \\
& + (r+\delta)^2(r+2\delta)[br-\delta(w-\tau\cdot\tilde{x}_1)]\} \\
& = \chi\cdot\left\{-\tilde{x}_2^2 2\tau\Psi_2(r+\delta)+\tilde{x}_2[2\Psi_2 w(r+\delta)-\tau r(r+2\delta)]+\left[r(r+2\delta)w+\delta^2(w-\tau\tilde{x}_1)\right]\right\}
\end{aligned}$$

et's work on the RHS of this expression $\chi\cdot\{-\tilde{x}_2^2 2\tau\Psi_2(r+\delta)+\tilde{x}_2[2\Psi_2 w(r+\delta)-\tau r(r+2\delta)]+[r(r+2\delta)w+\delta^2(w-\tau\tilde{x}_1)]\}$ where

$$\begin{aligned}
\chi &= (2\tilde{x}_2\Psi_2)^3 r+(2\tilde{x}_2\Psi_2)^2(r+\delta)[2(r+\delta)+r]+(2\tilde{x}_2\Psi_2)(r+2\delta)\left[2(r+\delta)^2+r^2\right]+r(r+\delta)(r+2\delta)^2 \\
\chi &= \tilde{x}_2^3\cdot 8\Psi_2^3 r+\tilde{x}_2^2\cdot 4\Psi_2^2(r+\delta)[2(r+\delta)+r]+\tilde{x}_2\cdot 2\Psi_2(r+2\delta)\left[2(r+\delta)^2+r^2\right]+r(r+\delta)(r+2\delta)^2
\end{aligned}$$

he product $\chi\cdot\{-\tilde{x}_2^2 2\tau\Psi_2(r+\delta)+\tilde{x}_2[2\Psi_2 w(r+\delta)-\tau r(r+2\delta)]+[r(r+2\delta)w+\delta^2(w-\tau\tilde{x}_1)]\}$ yields

$$\begin{aligned}
\text{HS} = & -\tilde{x}_2^5\cdot 16\Psi_2^4\tau r(r+\delta)+\tilde{x}_2^4 8\Psi_2^3 r[2\Psi_2 w(r+\delta)-\tau r(r+2\delta)]+\tilde{x}_2^3 8\Psi_2^3 r\left[rw(r+2\delta)+\delta^2(w-\tau\tilde{x}_1)\right] \\
& -\tilde{x}_2^4\cdot 8\Psi_2^3\tau(r+\delta)^2[2(r+\delta)+r]+\tilde{x}_2^3\cdot 4\Psi_2^2(r+\delta)[2(r+\delta)+r][2\Psi_2 w(r+\delta)-\tau r(r+2\delta)]+\tilde{x}_2^2\cdot 4\Psi_2^2(r+\delta)[2(r+\delta)+r]\left[rw(r+2\delta)+\delta^2(w-\tau\tilde{x}_1)\right] \\
& -\tilde{x}_2^3\cdot 4\Psi_2^2\tau(r+\delta)(r+2\delta)\left[2(r+\delta)^2+r^2\right]+\tilde{x}_2^2\cdot 2\Psi_2(r+2\delta)\left[2(r+\delta)^2+r^2\right][2\Psi_2 w(r+\delta)-\tau r(r+2\delta)]+\tilde{x}_2\cdot 2\Psi_2(r+2\delta)\left[2(r+\delta)^2+r^2\right][rw(r+2\delta)+\delta^2(w-\tau\tilde{x}_1)] \\
& -\tilde{x}_2^2\cdot 2\Psi_2\tau r(r+\delta)^2(r+2\delta)^2+\tilde{x}_2\cdot r(r+\delta)(r+2\delta)^2[2\Psi_2 w(r+\delta)-\tau r(r+2\delta)]+r(r+\delta)(r+2\delta)^2\left[r(r+2\delta)w+\delta^2(w-\tau\tilde{x}_1)\right]
\end{aligned}$$

Grouping $\tilde{x}_2^5, \tilde{x}_2^4, \tilde{x}_2^3, \tilde{x}_2^2, \tilde{x}_2$

$$\begin{aligned}
\text{RHS} = &- \tilde{x}_2^5 \cdot 16\Psi_2^4 \tau r \left(r+\delta\right) \\
&+ \tilde{x}_2^4 8\Psi_2^3 r \left[2\Psi_2 w \left(r+\delta\right) - \tau r \left(r+2\delta\right)\right] - \tilde{x}_2^4 \cdot 8\Psi_2^3 \tau \left(r+\delta\right)^2 \left[2\left(r+\delta\right)+r\right] \\
&+ \tilde{x}_2^3 8\Psi_2^3 r \left[rw\left(r+2\delta\right) + \delta^2\left(w-\tau\tilde{x}_1\right)\right] + \tilde{x}_2^3 \cdot 4\Psi_2^2 \left(r+\delta\right)\left[2\left(r+\delta\right)+r\right]\left[2\Psi_2 w\left(r+\delta\right) - \tau r\left(r+2\delta\right)\right] - \tilde{x}_2^3 \cdot 4\Psi_2^2 \tau \left(r+\delta\right)\left(r+2\delta\right)\left[2\left(r+\delta\right)^2 + r^2\right] \\
&+ \tilde{x}_2^2 \cdot 4\Psi_2^2 \left(r+\delta\right)\left[2\left(r+\delta\right)+r\right]\left[rw\left(r+2\delta\right)+\delta^2\left(w-\tau\tilde{x}_1\right)\right] + \tilde{x}_2^2 \cdot 2\Psi_2 \left(r+2\delta\right)\left[2\left(r+\delta\right)^2+r^2\right]\left[2\Psi_2 w\left(r+\delta\right)-\tau r\left(r+2\delta\right)\right] - \tilde{x}_2^2 \cdot 2\Psi_2 \tau r\left(r+\delta\right)^2\left(r+2\delta\right)^2 \\
&+ \tilde{x}_2 \cdot 2\Psi_2 \left(r+2\delta\right)\left[2\left(r+\delta\right)^2+r^2\right]\left[rw\left(r+2\delta\right)+\delta^2\left(w-\tau\tilde{x}_1\right)\right] + \tilde{x}_2 \cdot r\left(r+\delta\right)\left(r+2\delta\right)^2\left[2\Psi_2 w\left(r+\delta\right)-\tau r\left(r+2\delta\right)\right] \\
&+ r\left(r+\delta\right)\left(r+2\delta\right)^2\left[r\left(r+2\delta\right)w+\delta^2\left(w-\tau\tilde{x}_1\right)\right]
\end{aligned}$$

Factorizing $\tilde{x}_2^5, \tilde{x}_2^4, \tilde{x}_2^3, \tilde{x}_2^2, \tilde{x}_2$

$$\begin{aligned}
\text{RHS} = &- \tilde{x}_2^5 16\Psi_2^4 \tau r\left(r+\delta\right) \qquad (\text{H.28}) \\
&+ \tilde{x}_2^4 8\Psi_2^3 \left\{r\left[2\Psi_2 w\left(r+\delta\right)-\tau r\left(r+2\delta\right)\right] - \tau\left(r+\delta\right)^2\left[2\left(r+\delta\right)+r\right]\right\} \\
&+ \tilde{x}_2^3 4\Psi_2^2 \left\{2\Psi_2 r\left[rw\left(r+2\delta\right)+\delta^2\left(w-\tau\tilde{x}_1\right)\right] + \left(r+\delta\right)\left[2\left(r+\delta\right)+r\right]\left[2\Psi_2 w\left(r+\delta\right)-\tau r\left(r+2\delta\right)\right] - \tau\left(r+\delta\right)\left(r+2\delta\right)\left[2\left(r+\delta\right)^2+r^2\right]\right\} \\
&+ \tilde{x}_2^2 2\Psi_2 \left\{2\Psi_2\left(r+\delta\right)\left[2\left(r+\delta\right)+r\right]\left[rw\left(r+2\delta\right)+\delta^2\left(w-\tau\tilde{x}_1\right)\right] + \left(r+2\delta\right)\left[2\left(r+\delta\right)^2+r^2\right]\left[2\Psi_2 w\left(r+\delta\right)-\tau r\left(r+2\delta\right)\right] - \tau r\left(r+\delta\right)^2\left(r+2\delta\right)^2\right\} \\
&+ \tilde{x}_2\left(r+2\delta\right)\left\{2\Psi_2\left[2\left(r+\delta\right)^2+r^2\right]\left[rw\left(r+2\delta\right)+\delta^2\left(w-\tau\tilde{x}_1\right)\right] + r\left(r+\delta\right)\left(r+2\delta\right)\left[2\Psi_2 w\left(r+\delta\right)-\tau r\left(r+2\delta\right)\right]\right\} \\
&+ r\left(r+\delta\right)\left(r+2\delta\right)^2\left[r\left(r+2\delta\right)w+\delta^2\left(w-\tau\tilde{x}_1\right)\right]
\end{aligned}$$

Let's work on the LHS of this expression

$$\begin{aligned}
\text{LHS} =& r\left(r+2\delta+2\tilde{x}_2\Psi_2\right)\left\{-\tilde{x}_2^4 4\tau\Psi_2^3\left(r+\delta\right)\right.\\
&+\tilde{x}_2^3 2\Psi_2^2\left\{4\Psi_2 w\left(r+\delta\right)-\tau\left[\left(r+\delta\right)^2+r\left(r+2\delta\right)\right]\right\}\\
&+\tilde{x}_2^2\Psi_2\left\{4\Psi_2 b\left(r+\delta\right)^2+4\Psi_2 w\left[\left(r+\delta\right)^2+r\left(r+2\delta\right)\right]-\tau r\left(r+\delta\right)\left(r+2\delta\right)-4\Psi_2 r\delta\left(w-\tau\cdot\tilde{x}_1\right)\right\}\\
&+\tilde{x}_2 2\Psi_2\left(r+\delta\right)\left\{b\left[\left(r+\delta\right)^2+r\left(r+2\delta\right)\right]+rw\left(r+2\delta\right)-2\delta\left(r+\delta\right)\left(w-\tau\cdot\tilde{x}_1\right)\right\}\\
&\left.+\left(r+\delta\right)^2\left(r+2\delta\right)\left[br-\delta\left(w-\tau\cdot\tilde{x}_1\right)\right]\right\}
\end{aligned}$$

Multiplying $r\left(r+2\delta+2\tilde{x}_2\Psi_2\right)=r\left(r+2\delta\right)+r2\tilde{x}_2\Psi_2$

$$\begin{aligned}
\text{LHS} =& -\tilde{x}_2^4 4\Psi_2^3\tau r\left(r+\delta\right)\left(r+2\delta\right)\\
&+\tilde{x}_2^3 2\Psi_2^2 r\left(r+2\delta\right)\left\{4\Psi_2 w\left(r+\delta\right)-\tau\left[\left(r+\delta\right)^2+r\left(r+2\delta\right)\right]\right\}\\
&+\tilde{x}_2^2\Psi_2 r\left(r+2\delta\right)\left\{4\Psi_2 b\left(r+\delta\right)^2+4\Psi_2 w\left[\left(r+\delta\right)^2+r\left(r+2\delta\right)\right]-\tau r\left(r+\delta\right)\left(r+2\delta\right)-4\Psi_2 r\delta\left(w-\tau\cdot\tilde{x}_1\right)\right\}\\
&+\tilde{x}_2 2\Psi_2 r\left(r+\delta\right)\left(r+2\delta\right)\left\{b\left[\left(r+\delta\right)^2+r\left(r+2\delta\right)\right]+rw\left(r+2\delta\right)-2\delta\left(r+\delta\right)\left(w-\tau\cdot\tilde{x}_1\right)\right\}\\
&+r\left(r+\delta\right)^2\left(r+2\delta\right)^2\left[br-\delta\left(w-\tau\cdot\tilde{x}_1\right)\right]\\
&-\tilde{x}_2^5 8\Psi_2^4\tau r\left(r+\delta\right)\\
&+\tilde{x}_2^4 4\Psi_2^3 r\left\{4\Psi_2 w\left(r+\delta\right)-\tau\left[\left(r+\delta\right)^2+r\left(r+2\delta\right)\right]\right\}\\
&+\tilde{x}_2^3 2\Psi_2^2 r\left\{4\Psi_2 b\left(r+\delta\right)^2+4\Psi_2 w\left[\left(r+\delta\right)^2+r\left(r+2\delta\right)\right]-\tau r\left(r+\delta\right)\left(r+2\delta\right)-4\Psi_2 r\delta\left(w-\tau\cdot\tilde{x}_1\right)\right\}\\
&+\tilde{x}_2^2 4\Psi_2^2 r\left(r+\delta\right)\left\{b\left[\left(r+\delta\right)^2+r\left(r+2\delta\right)\right]+rw\left(r+2\delta\right)-2\delta\left(r+\delta\right)\left(w-\tau\cdot\tilde{x}_1\right)\right\}\\
&+2\tilde{x}_2\Psi_2 r\left(r+\delta\right)^2\left(r+2\delta\right)\left[br-\delta\left(w-\tau\cdot\tilde{x}_1\right)\right]
\end{aligned}$$

Grouping $\tilde{x}_2^5, \tilde{x}_2^4, \tilde{x}_2^3, \tilde{x}_2^2, \tilde{x}_2$

$$\begin{aligned}
\text{LHS} = &- \tilde{x}_2^5 \Psi_2^4 8 r\tau (r+\delta) \\
&+ \tilde{x}_2^4 4\Psi_2^3 r \left\{4\Psi_2 w (r+\delta) - \tau \left[(r+\delta)^2 + r(r+2\delta)\right]\right\} - \tilde{x}_2^4 4\tau\Psi_2^3 r (r+\delta)(r+2\delta) \\
&+ \tilde{x}_2^3 2\Psi_2^2 r (r+2\delta) \left\{4\Psi_2 w (r+\delta) - \tau \left[(r+\delta)^2 + r(r+2\delta)\right]\right\} \\
&+ \tilde{x}_2^3 2\Psi_2^2 r \left\{4\Psi_2 b (r+\delta)^2 + 4\Psi_2 w \left[(r+\delta)^2 + r(r+2\delta)\right] - \tau r (r+\delta)(r+2\delta) - 4\Psi_2 r\delta (w - \tau\cdot\tilde{x}_1)\right\} \\
&+ \tilde{x}_2^2 \Psi_2 r (r+2\delta) \left\{4\Psi_2 b (r+\delta)^2 + 4\Psi_2 w \left[(r+\delta)^2 + r(r+2\delta)\right] - \tau r (r+\delta)(r+2\delta) - 4\Psi_2 r\delta (w - \tau\cdot\tilde{x}_1)\right\} \\
&+ \tilde{x}_2^2 4\Psi_2^2 r (r+\delta) \left\{b\left[(r+\delta)^2 + r(r+2\delta)\right] + rw(r+2\delta) - 2\delta(r+\delta)(w-\tau\cdot\tilde{x}_1)\right\} \\
&+ \tilde{x}_2 2\Psi_2 r (r+\delta)(r+2\delta) \left\{b\left[(r+\delta)^2 + r(r+2\delta)\right] + rw(r+2\delta) - 2\delta(r+\delta)(w-\tau\cdot\tilde{x}_1)\right\} + 2\tilde{x}_2\Psi_2 r (r+\delta)^2 (r+2\delta)\left[br - \delta(w-\tau\cdot\tilde{x}_1)\right] \\
&+ r(r+\delta)^2(r+2\delta)^2\left[br - \delta(w-\tau\cdot\tilde{x}_1)\right]
\end{aligned}$$

Factorizing $\tilde{x}_2^5, \tilde{x}_2^4, \tilde{x}_2^3, \tilde{x}_2^2, \tilde{x}_2$

$$\begin{aligned}
\text{LHS} = &- \tilde{x}_2^5 8\Psi_2^4 \tau r (r+\delta) \\
&+ \tilde{x}_2^4 4\Psi_2^3 r \left\{4\Psi_2 w (r+\delta) - \tau\left[(r+\delta)^2 + r(r+2\delta)\right] - \tau(r+\delta)(r+2\delta)\right\} \\
&+ \tilde{x}_2^3 2\Psi_2^2 r \left\langle 4\Psi_2 w (r+\delta)(r+2\delta) - \tau(r+2\delta)\left[(r+\delta)^2 + r(r+2\delta)\right] + 4\Psi_2 b (r+\delta)^2 + 4\Psi_2 w\left[(r+\delta)^2 + r(r+2\delta)\right] - \tau r (r+\delta)(r+2\delta) - 4\Psi_2 r\delta(w-\tau\cdot\tilde{x}_1)\right\rangle \\
&+ \tilde{x}_2^2 \Psi_2 r \left\langle 4\Psi_2 b (r+\delta)^2 (r+2\delta) + 4\Psi_2 w (r+2\delta)\left[(r+\delta)^2 + r(r+2\delta)\right] - \tau r (r+\delta)(r+2\delta)^2 - 4\Psi_2 r\delta(r+2\delta)(w-\tau\cdot\tilde{x}_1) + 4\Psi_2(r+\delta)\left\{b\left[(r+\delta)^2 + r(r+2\delta)\right] + rw(r+2\delta) - 2\delta(r+\delta)(w-\tau\cdot\tilde{x}_1)\right\}\right\rangle \\
&+ \tilde{x}_2 2\Psi_2 r (r+\delta)(r+2\delta)\left\{b\left[(r+\delta)^2 + r(r+2\delta)\right] + rw(r+2\delta) - 2\delta(r+\delta)(w-\tau\cdot\tilde{x}_1) + (r+\delta)\left[br - \delta(w-\tau\cdot\tilde{x}_1)\right]\right\} \\
&+ r(r+\delta)^2(r+2\delta)^2\left[br - \delta(w-\tau\cdot\tilde{x}_1)\right]
\end{aligned} \tag{H.29}$$

Putting (H.28) and (H.29) together, LHS = RHS, yields

$$
\begin{aligned}
&-\tilde{x}_2^5 8\Psi_2^4 \tau r\,(r+\delta)\\
&+\tilde{x}_2^4 4\Psi_2^3 r\left\{4\Psi_2 w\,(r+\delta)-\tau\left[(r+\delta)^2+r\,(r+2\delta)\right]-\tau\,(r+\delta)\,(r+2\delta)\right\}\\
&+\tilde{x}_2^3 2\Psi_2^2 r\left\langle 4\Psi_2 w\,(r+\delta)\,(r+2\delta)-\tau\,(r+2\delta)\left[(r+\delta)^2+r\,(r+2\delta)\right]+4\Psi_2 b\,(r+\delta)^2+4\Psi_2 w\left[(r+\delta)^2+r\,(r+2\delta)\right]-\tau r\,(r+\delta)\,(r+2\delta)-4\Psi_2 r\delta\,(w-\tau\cdot\tilde{x}_1)\right\rangle\\
&+\tilde{x}_2^2\Psi_2 r\left\langle\left\{4\Psi_2 b\,(r+\delta)^2\,(r+2\delta)+4\Psi_2 w\,(r+2\delta)\left[(r+\delta)^2+r\,(r+2\delta)\right]-\tau r\,(r+\delta)\,(r+2\delta)^2-4\Psi_2 r\delta\,(r+2\delta)\,(w-\tau\cdot\tilde{x}_1)\right\}+4\Psi_2\,(r+\delta)\left\{b\left[(r+\delta)^2+r\,(r+2\delta)\right]+rw\,(r+2\delta)-2\delta\,(r+\delta)\,(w-\tau\cdot\tilde{x}_1)\right\}\right\rangle\\
&+\tilde{x}_2 2\Psi_2 r\,(r+\delta)\,(r+2\delta)\left\{b\left[(r+\delta)^2+r\,(r+2\delta)\right]+rw\,(r+2\delta)-2\delta\,(r+\delta)\,(w-\tau\cdot\tilde{x}_1)+(r+\delta)\left[br-\delta\,(w-\tau\cdot\tilde{x}_1)\right]\right\}\\
&+r\,(r+\delta)^2\,(r+2\delta)^2\left[br-\delta\,(w-\tau\cdot\tilde{x}_1)\right]\\
=&-\tilde{x}_2^5 16\Psi_2^4\tau r\,(r+\delta)\\
&+\tilde{x}_2^4 8\Psi_2^3\left\{r\left[2\Psi_2 w\,(r+\delta)-\tau r\,(r+2\delta)\right]-\tau\,(r+\delta)^2\left[2\,(r+\delta)+r\right]\right\}\\
&+\tilde{x}_2^3 4\Psi_2^2\left\{2\Psi_2 r\left[rw\,(r+2\delta)+\delta^2\,(w-\tau\tilde{x}_1)\right]+(r+\delta)\left[2\,(r+\delta)+r\right]\left[2\Psi_2 w\,(r+\delta)-\tau r\,(r+2\delta)\right]-\tau\,(r+\delta)\,(r+2\delta)\left[2\,(r+\delta)^2+r^2\right]\right\}\\
&+\tilde{x}_2^2 2\Psi_2\left\{2\Psi_2\,(r+\delta)\left[2\,(r+\delta)+r\right]\left[rw\,(r+2\delta)+\delta^2\,(w-\tau\tilde{x}_1)\right]+(r+2\delta)\left[2\,(r+\delta)^2+r^2\right]\left[2\Psi_2 w\,(r+\delta)-\tau r\,(r+2\delta)\right]-\tau r\,(r+\delta)^2\,(r+2\delta)^2\right\}\\
&+\tilde{x}_2\,(r+2\delta)\left\{2\Psi_2\left[2\,(r+\delta)^2+r^2\right]\left[rw\,(r+2\delta)+\delta^2\,(w-\tau\tilde{x}_1)\right]+r\,(r+\delta)\,(r+2\delta)\left[2\Psi_2 w\,(r+\delta)-\tau r\,(r+2\delta)\right]\right\}\\
&+r\,(r+\delta)\,(r+2\delta)^2\left[r\,(r+2\delta)\,w+\delta^2\,(w-\tau\tilde{x}_1)\right]
\end{aligned}
$$

After some simplifications, $LHS\text{-}RHS = 0$

$$
\begin{aligned}
&+\tilde{x}_2^5 8\Psi_2^4\tau r\,(r+\delta)\,OK\\
&+\tilde{x}_2^4 4\Psi_2^3\left\{4\Psi_2 rw\,(r+\delta)-\tau r\left[(r+\delta)^2+r\,(r+2\delta)-(r+\delta)\,(r+2\delta)\right]-4\Psi_2 rw\,(r+\delta)+2\tau r^2\,(r+2\delta)+2\tau\,(r+\delta)^2\left[2\,(r+\delta)+r\right]\right\}OK\\
&+\tilde{x}_2^3 2\Psi_2^2\left\{4\Psi_2 rw\,(r+\delta)\,(r+2\delta)-\tau r\,(r+2\delta)\left[(r+\delta)^2+r\,(r+2\delta)\right]+4\Psi_2 br\,(r+\delta)^2+4\Psi_2 rw\left[(r+\delta)^2+r\,(r+2\delta)\right]-\tau r^2\,(r+\delta)\,(r+2\delta)-4\Psi_2 r^2\delta\,(w-\tau\tilde{x}_1)\right.\\
&\left.-4\Psi_2 r\left[rw\,(r+2\delta)+\delta^2\,(w-\tau\tilde{x}_1)\right]-2\,(r+\delta)\left[2\,(r+\delta)+r\right]\left[2\Psi_2 w\,(r+\delta)-\tau r\,(r+2\delta)\right]+2\tau\,(r+\delta)\,(r+2\delta)\left[2\,(r+\delta)^2+r^2\right]\right\}OK\\
&+\tilde{x}_2^2\Psi_2\left\{4\Psi_2 br\,(r+\delta)^2\,(r+2\delta)+4\Psi_2 rw\,(r+2\delta)\left[(r+\delta)^2+r\,(r+2\delta)\right]-\tau r^2\,(r+\delta)\,(r+2\delta)^2-4\Psi_2 r^2\delta\,(r+2\delta)\,(w-\tau\tilde{x}_1)+4\Psi_2 r\,(r+\delta)\left\{b\left[(r+\delta)^2+r\,(r+2\delta)\right]+rw\,(r+2\delta)-2\delta\,(r+\delta)\,(w-\tau\cdot\tilde{x}_1)\right\}\right.\\
&\left.-2\left\{2\Psi_2\,(r+\delta)\left[2\,(r+\delta)+r\right]\left[rw\,(r+2\delta)+\delta^2\,(w-\tau\tilde{x}_1)\right]+(r+2\delta)\left[2\,(r+\delta)^2+r^2\right]\left[2\Psi_2 w\,(r+\delta)-\tau r\,(r+2\delta)\right]-\tau r\,(r+\delta)^2\,(r+2\delta)^2\right\}\right\}OK\\
&+\tilde{x}_2\,(r+2\delta)\left\{+2\Psi_2 r\,(r+\delta)\left\{b\left[(r+\delta)^2+r\,(r+2\delta)\right]+rw\,(r+2\delta)-2\delta\,(r+\delta)\,(w-\tau\cdot\tilde{x}_1)+(r+\delta)\left[br-\delta\,(w-\tau\cdot\tilde{x}_1)\right]\right\}\right.\\
&\left.-2\Psi_2\left[2\,(r+\delta)^2+r^2\right]\left[rw\,(r+2\delta)+\delta^2\,(w-\tau\tilde{x}_1)\right]-r\,(r+\delta)\,(r+2\delta)\left[2\Psi_2 w\,(r+\delta)+\tau r\,(r+2\delta)\right]\right\}OK\\
&+r\,(r+\delta)\,(r+2\delta)^2\left\{(r+\delta)\left[br-\delta\,(w-\tau\cdot\tilde{x}_1)\right]-rw\,(r+2\delta)-\delta^2\,(w-\tau\tilde{x}_1)\right\}=0K
\end{aligned}
$$

we have the implicit function:

$$
\beta_5\cdot\tilde{x}_2^5+\beta_4\cdot\tilde{x}_2^4+\beta_3\cdot\tilde{x}_2^3+\beta_2\cdot\tilde{x}_2^2+\beta_1\cdot\tilde{x}_2+\beta_0=0 \tag{H.30}
$$

where:

$$\beta_5 = 8\Psi_2^4\tau r(r+\delta)$$

$$\beta_4 = 4\Psi_2^3\left\{\tau r\left[3\left(r^2+\delta^2\right)+7r\delta\right]+4\tau\left(r+\delta\right)^3\right\}$$

$$\beta_3 = 2\Psi_2^2\Bigg\{4\Psi_2 rw\left(r+\delta\right)\left(r+2\delta\right)+4\Psi_2 br\left(r+\delta\right)^2+3\tau r\left(r+\delta\right)^2\left(r+2\delta\right)+\tau r^2\left(r+2\delta\right)\left(2r+\delta\right)$$
$$+4\tau\left(r+\delta\right)^3\left(r+2\delta\right)-4\Psi_2 r\delta\left(r+\delta\right)\left(w-\tau\tilde{x}_1\right)-8\Psi_2 w\left(r+\delta\right)^3\Bigg\}$$

$$\beta_2 = \Psi_2\Bigg\{4\Psi_2 br^2\left(r+\delta\right)\left(r+2\delta\right)+6\tau r\left(r+\delta\right)^2\left(r+2\delta\right)^2+\tau r^3\left(r+2\delta\right)^2$$
$$-\tau r^2\delta\left(r+2\delta\right)^2-4\Psi_2 r\left(r+\delta\right)^2\left(r+2\delta\right)\left(w-b\right)-4\Psi_2 r^3 w\left(r+2\delta\right)-4\Psi_2 r\left(r+\delta\right)^3\left(2w-b\right)$$
$$-16\Psi_2 w\delta\left(r+\delta\right)^3-4\Psi_2 r\delta\left(r+\delta\right)\left(r+2\delta\right)\left(w-\tau\tilde{x}_1\right)-8\Psi_2\delta\left(r+\delta\right)^3\left(w-\tau\tilde{x}_1\right)\Bigg\}$$

$$\beta_1 = \left(r+2\delta\right)\Bigg\{2\Psi_2 br^2\left(r+\delta\right)\left(r+2\delta\right)+2\Psi_2 br\left(r+\delta\right)^3+2\Psi_2 r^2 w\delta\left(r+2\delta\right)+2\Psi_2 br^2\left(r+\delta\right)^2+\tau r^2\left(r+\delta\right)\left(r+2\delta\right)^2$$
$$-6\Psi_2 r\delta\left(r+\delta\right)^2\left(w-\tau\tilde{x}_1\right)-6\Psi_2 rw\left(r+\delta\right)^2\left(r+2\delta\right)-4\Psi_2\delta^2\left(r+\delta\right)^2\left(w-\tau\tilde{x}_1\right)-2\Psi_2 r^2\delta^2\left(w-\tau\tilde{x}_1\right)\Bigg\}$$

$$\beta_0 = -r\left(r+\delta\right)\left(r+2\delta\right)^2\left[r^2\left(w-b\right)+r\delta\left(2w-b\right)+\delta\left(r+2\delta\right)\left(w-\tau\tilde{x}_1\right)\right]$$

Detailed computations of the quintic function for $\tilde{x}_2$

1. The coefficient of $\tilde{x}_2^4$, β_4

$$+\tilde{x}_2^4 4\Psi_2^3\left\{4\Psi_2 rw\left(r+\delta\right)-\tau r\left[\left(r+\delta\right)^2+r\left(r+2\delta\right)-\left(r+\delta\right)\left(r+2\delta\right)\right]-4\Psi_2 rw\left(r+\delta\right)+2\tau r^2\left(r+2\delta\right)+2\tau\left(r+\delta\right)^2\left[2\left(r+\delta\right)+r\right]\right\}$$

$$= 4\Psi_2 rw\left(r+\delta\right)-\tau r\left[\left(r+\delta\right)^2+r\left(r+2\delta\right)-\left(r+\delta\right)\left(r+2\delta\right)\right]-4\Psi_2 rw\left(r+\delta\right)+2\tau r^2\left(r+2\delta\right)+2\tau\left(r+\delta\right)^2\left[2\left(r+\delta\right)+r\right]$$
$$= 2\tau r^2\left(r+2\delta\right)-\tau r\left[\left(r+\delta\right)^2+r\left(r+2\delta\right)-\left(r+\delta\right)\left(r+2\delta\right)\right]+2\tau r\left(r+\delta\right)^2+4\tau\left(r+\delta\right)^3$$

Factorizing τr

$$
\begin{aligned}
&= \tau r\left[2r(r+2\delta)-(r+\delta)^2-r(r+2\delta)+(r+\delta)(r+2\delta)+2(r+\delta)^2\right]+4\tau(r+\delta)^3 \\
&= \tau r\left[2r(r+2\delta)+(r+\delta)^2-r(r+2\delta)+(r+\delta)(r+2\delta)\right]+4\tau(r+\delta)^3 \\
&= \tau r\left[2r^2+4r\delta+r^2+2r\delta+\delta^2-r^2-2r\delta+r^2+2r\delta+r\delta+2\delta^2\right]+4\tau(r+\delta)^3 \\
&= \tau r\left[3r^2+7r\delta+3\delta^2\right]+4\tau(r+\delta)^3 \\
&= \tau r\left[3\left(r^2+\delta^2\right)+7r\delta\right]+4\tau(r+\delta)^3
\end{aligned}
$$

2. The coefficient of $\tilde{x}_2^3$, β_3

$$
\begin{aligned}
&= 4\Psi_2 rw(r+\delta)(r+2\delta)-\tau r(r+2\delta)\left[(r+\delta)^2+r(r+2\delta)\right]+4\Psi_2 br(r+\delta)^2+4\Psi_2 rw\left[(r+\delta)^2+r(r+2\delta)\right]-\tau r^2(r+\delta)(r+2\delta)-4\Psi_2 r^2\delta(w-\tau\tilde{x}_1) \\
&-4\Psi_2 r\left[rw(r+2\delta)+\delta^2(w-\tau\tilde{x}_1)\right]-2(r+\delta)[2(r+\delta)+r][2\Psi_2 w(r+\delta)-\tau r(r+2\delta)]+2\tau(r+\delta)(r+2\delta)\left[2(r+\delta)^2+r^2\right] \\
&= 4\Psi_2 rw(r+\delta)(r+2\delta)-\tau r(r+\delta)^2(r+2\delta)-\tau r^2(r+2\delta)^2+4\Psi_2 br(r+\delta)^2+4\Psi_2 rw(r+\delta)^2+4\Psi_2 r^2 w(r+2\delta)-\tau r^2(r+\delta)(r+2\delta)-4\Psi_2 r^2\delta(w-\tau\tilde{x}_1) \\
&-4\Psi_2 r^2 w(r+2\delta)-4\Psi_2 r\delta^2(w-\tau\tilde{x}_1)-\left[4(r+\delta)^2+2r(r+\delta)\right][2\Psi_2 w(r+\delta)-\tau r(r+2\delta)]+4\tau(r+\delta)^3(r+2\delta)+2\tau r^2(r+\delta)(r+2\delta) \\
&= 4\Psi_2 rw(r+\delta)(r+2\delta)-\tau r(r+\delta)^2(r+2\delta)-\tau r^2(r+2\delta)^2+4\Psi_2 br(r+\delta)^2+4\Psi_2 rw(r+\delta)^2+4\Psi_2 r^2 w(r+2\delta)-\tau r^2(r+\delta)(r+2\delta)-4\Psi_2 r^2\delta(w-\tau\tilde{x}_1) \\
&-4\Psi_2 r^2 w(r+2\delta)-4\Psi_2 r\delta^2(w-\tau\tilde{x}_1)-8\Psi_2 w(r+\delta)^3+4\tau r(r+\delta)^2(r+2\delta)-4\Psi_2 rw(r+\delta)^2+2\tau r^2(r+\delta)(r+2\delta)+4\tau(r+\delta)^3(r+2\delta)+2\tau r^2(r+\delta)(r+2\delta) \\
&= 4\Psi_2 rw(r+\delta)(r+2\delta)+4\Psi_2 br(r+\delta)^2+4\Psi_2 rw(r+\delta)^2+4\Psi_2 r^2 w(r+2\delta)+4\tau r(r+\delta)^2(r+2\delta)+2\tau r^2(r+\delta)(r+2\delta)+4\tau(r+\delta)^3(r+2\delta)+2\tau r^2(r+\delta)(r+2\delta) \\
&-\tau r(r+\delta)^2(r+2\delta)-\tau r^2(r+2\delta)^2-\tau r^2(r+\delta)(r+2\delta)-4\Psi_2 r^2\delta(w-\tau\tilde{x}_1)-4\Psi_2 r\delta^2(w-\tau\tilde{x}_1)-4\Psi_2 r^2 w(r+2\delta)-8\Psi_2 w(r+\delta)^3-4\Psi_2 rw(r+\delta)^2
\end{aligned}
$$

Factorizing

$$
\begin{aligned}
=&4\Psi_2 rw\left(r+\delta\right)\left(r+2\delta\right)+4\Psi_2 br\left(r+\delta\right)^2+4\Psi_2 r^2 w\left(r+2\delta\right)+3\tau r\left(r+\delta\right)^2\left(r+2\delta\right)+3\tau r^2\left(r+\delta\right)\left(r+2\delta\right)+4\tau\left(r+\delta\right)^3\left(r+2\delta\right)\\
&-\tau r^2\left(r+2\delta\right)^2-4\Psi_2 r^2\delta\left(w-\tau\tilde{x}_1\right)-4\Psi_2 r\delta^2\left(w-\tau\tilde{x}_1\right)-4\Psi_2 r^2 w\left(r+2\delta\right)-8\Psi_2 w\left(r+\delta\right)^3\\
=&4\Psi_2 rw\left(r+\delta\right)\left(r+2\delta\right)+4\Psi_2 br\left(r+\delta\right)^2+4\Psi_2 r^2 w\left(r+2\delta\right)+3\tau r\left(r+\delta\right)^2\left(r+2\delta\right)+3\tau r^2\left(r+\delta\right)\left(r+2\delta\right)+4\tau\left(r+\delta\right)^3\left(r+2\delta\right)\\
&-\tau r^2\left(r+2\delta\right)^2-4\Psi_2 r\delta\left(r+\delta\right)\left(w-\tau\tilde{x}_1\right)-4\Psi_2 r^2 w\left(r+2\delta\right)-8\Psi_2 w\left(r+\delta\right)^3\\
=&4\Psi_2 rw\left(r+2\delta\right)\left(r+\delta-r\right)+4\Psi_2 br\left(r+\delta\right)^2+4\Psi_2 r^2 w\left(r+2\delta\right)+3\tau r\left(r+\delta\right)^2\left(r+2\delta\right)+\tau r^2\left(r+2\delta\right)\left[3r+3\delta-r-2\delta\right]\\
&+4\tau\left(r+\delta\right)^3\left(r+2\delta\right)-4\Psi_2 r\delta\left(r+\delta\right)\left(w-\tau\tilde{x}_1\right)-8\Psi_2 w\left(r+\delta\right)^3\\
=&4\Psi_2 rw\delta\left(r+2\delta\right)+4\Psi_2 br\left(r+\delta\right)^2+4\Psi_2 r^2 w\left(r+2\delta\right)+3\tau r\left(r+\delta\right)^2\left(r+2\delta\right)+\tau r^2\left(r+2\delta\right)\left(2r+\delta\right)\\
&+4\tau\left(r+\delta\right)^3\left(r+2\delta\right)-4\Psi_2 r\delta\left(r+\delta\right)\left(w-\tau\tilde{x}_1\right)-8\Psi_2 w\left(r+\delta\right)^3\\
&=4\Psi_2 rw\left(r+\delta\right)\left(r+2\delta\right)+4\Psi_2 br\left(r+\delta\right)^2+3\tau r\left(r+\delta\right)^2\left(r+2\delta\right)+\tau r^2\left(r+2\delta\right)\left(2r+\delta\right)\\
&+4\tau\left(r+\delta\right)^3\left(r+2\delta\right)-4\Psi_2 r\delta\left(r+\delta\right)\left(w-\tau\tilde{x}_1\right)-8\Psi_2 w\left(r+\delta\right)^3 OK\ sofar
\end{aligned}
$$

3. The coefficient of $\tilde{x}_2^2$, , β_2

$$
\begin{aligned}
=&4\Psi_2 br\left(r+\delta\right)^2\left(r+2\delta\right)+4\Psi_2 rw\left(r+2\delta\right)\left[\left(r+\delta\right)^2+r\left(r+2\delta\right)\right]-\tau r^2\left(r+\delta\right)\left(r+2\delta\right)^2-4\Psi_2 r^2\delta\left(r+2\delta\right)\left(w-\tau\tilde{x}_1\right)+4\Psi_2 r\left(r+\delta\right)\left\{b\left[\left(r+\delta\right)^2+r\left(r+2\delta\right)\right]+rw\left(r+2\delta\right)-2\delta\left(r+\delta\right)\left(w-\tau\cdot\tilde{x}_1\right)\right\}\\
&-4\Psi_2\left(r+\delta\right)\left[2\left(r+\delta\right)+r\right]\left[rw\left(r+2\delta\right)+\delta^2\left(w-\tau\tilde{x}_1\right)\right]-2\left(r+2\delta\right)\left[2\left(r+\delta\right)^2+r^2\right]\left[2\Psi_2 w\left(r+\delta\right)-\tau r\left(r+2\delta\right)\right]+2\tau r\left(r+\delta\right)^2\left(r+2\delta\right)^2 OK\\
=&4\Psi_2 br\left(r+\delta\right)^2\left(r+2\delta\right)+4\Psi_2 r^2 w\left(r+2\delta\right)^2+4\Psi_2 rw\left(r+\delta\right)^2\left(r+2\delta\right)-\tau r^2\left(r+\delta\right)\left(r+2\delta\right)^2-4\Psi_2 r^2\delta\left(r+2\delta\right)\left(w-\tau\tilde{x}_1\right)+4\Psi_2 r^2 b\left(r+\delta\right)\left(r+2\delta\right)+4\Psi_2 rb\left(r+\delta\right)^3+4\Psi_2 r^2 w\left(r+\delta\right)\left(r+2\delta\right)-8\Psi_2 r\delta\left(r+\delta\right)^2\left(w-\tau\cdot\tilde{x}_1\right)\\
&-4\Psi_2\left(r+\delta\right)\left[2\left(r+\delta\right)+r\right]\left[rw\left(r+2\delta\right)+\delta^2\left(w-\tau\tilde{x}_1\right)\right]-2\left(r+2\delta\right)\left[2\left(r+\delta\right)^2+r^2\right]\left[2\Psi_2 w\left(r+\delta\right)-\tau r\left(r+2\delta\right)\right]+2\tau r\left(r+\delta\right)^2\left(r+2\delta\right)^2 OK\\
=&4\Psi_2 br\left(r+\delta\right)^2\left(r+2\delta\right)+4\Psi_2 r^2 w\left(r+2\delta\right)^2+4\Psi_2 rw\left(r+\delta\right)^2\left(r+2\delta\right)-\tau r^2\left(r+\delta\right)\left(r+2\delta\right)^2-4\Psi_2 r^2\delta\left(r+2\delta\right)\left(w-\tau\tilde{x}_1\right)+4\Psi_2 r^2 b\left(r+\delta\right)\left(r+2\delta\right)+4\Psi_2 rb\left(r+\delta\right)^3+4\Psi_2 r^2 w\left(r+\delta\right)\left(r+2\delta\right)-8\Psi_2 r\delta\left(r+\delta\right)^2\left(w-\tau\cdot\tilde{x}_1\right)\\
-&\left[8\Psi_2\left(r+\delta\right)^2+4\Psi_2 r\left(r+\delta\right)\right]\left[rw\left(r+2\delta\right)+\delta^2\left(w-\tau\tilde{x}_1\right)\right]-\left[4\left(r+\delta\right)^2\left(r+2\delta\right)+2r^2\left(r+2\delta\right)\right]\left[2\Psi_2 w\left(r+\delta\right)-\tau r\left(r+2\delta\right)\right]+2\tau r\left(r+\delta\right)^2\left(r+2\delta\right)^2 OK
\end{aligned}
$$

$$
\begin{aligned}
=&4\Psi_2 br(r+\delta)^2(r+2\delta)+4\Psi_2 r^2 w(r+2\delta)^2+4\Psi_2 rw(r+\delta)^2(r+2\delta)-\tau r^2(r+\delta)(r+2\delta)^2\\
&-4\Psi_2 r^2\delta(r+2\delta)(w-\tau\tilde{x}_1)+4\Psi_2 r^2 b(r+\delta)(r+2\delta)+4\Psi_2 rb(r+\delta)^3+4\Psi_2 r^2 w(r+\delta)(r+2\delta)-8\Psi_2 r\delta(r+\delta)^2(w-\tau\tilde{x}_1)\\
-&8\Psi_2 rw(r+\delta)^2(r+2\delta)-8\Psi_2\delta^2(r+\delta)^2(w-\tau\tilde{x}_1)-4\Psi_2 r^2 w(r+\delta)(r+2\delta)-4\Psi_2 r\delta^2(r+\delta)(w-\tau\tilde{x}_1)\\
&-8\Psi_2 w(r+\delta)^3(r+2\delta)+4\tau r(r+\delta)^2(r+2\delta)^2-4\Psi_2 r^2 w(r+\delta)(r+2\delta)+2\tau r^3(r+2\delta)^2+2\tau r(r+\delta)^2(r+2\delta)^2\,OK\\
=&4\Psi_2 br(r+\delta)^2(r+2\delta)+4\Psi_2 r^2 w(r+2\delta)^2+4\Psi_2 r^2 b(r+\delta)(r+2\delta)+4\Psi_2 rb(r+\delta)^3+6\tau r(r+\delta)^2(r+2\delta)^2+2\tau r^3(r+2\delta)^2\\
&-\tau r^2(r+\delta)(r+2\delta)^2-4\Psi_2 r^2\delta(r+2\delta)(w-\tau\tilde{x}_1)-8\Psi_2\delta(r+\delta)^3(w-\tau\tilde{x}_1)-4\Psi_2 r^2 w(r+\delta)(r+2\delta)\\
&-4\Psi_2 r\delta^2(r+\delta)(w-\tau\tilde{x}_1)-4\Psi_2 rw(r+\delta)^2(r+2\delta)-8\Psi_2 w(r+\delta)^3(r+2\delta)\\
=&4\Psi_2 br(r+\delta)^2(r+2\delta)+4\Psi_2 r^2 w(r+2\delta)^2+4\Psi_2 r^2 b(r+\delta)(r+2\delta)+4\Psi_2 rb(r+\delta)^3+6\tau r(r+\delta)^2(r+2\delta)^2+2\tau r^3(r+2\delta)^2\\
&-\tau r^2(r+\delta)(r+2\delta)^2-4\Psi_2 r^2\delta(r+2\delta)(w-\tau\tilde{x}_1)-8\Psi_2\delta(r+\delta)^3(w-\tau\tilde{x}_1)-4\Psi_2 r^2 w(r+\delta)(r+2\delta)\\
&-4\Psi_2 r\delta^2(r+\delta)(w-\tau\tilde{x}_1)-4\Psi_2 rw(r+\delta)^2(r+2\delta)-8\Psi_2 w(r+\delta)^3(r+2\delta)\\
=&4\Psi_2 br^2(r+\delta)(r+2\delta)+6\tau r(r+\delta)^2(r+2\delta)^2+\tau r^3(r+2\delta)^2\\
&-\tau r^2\delta(r+2\delta)^2-4\Psi_2 r(r+\delta)^2(r+2\delta)(w-b)-4\Psi_2 r^3 w(r+2\delta)-4\Psi_2 r(r+\delta)^3(2w-b)\\
&-16\Psi_2 w\delta(r+\delta)^3-4\Psi_2 r\delta(r+\delta)(r+2\delta)(w-\tau\tilde{x}_1)-8\Psi_2\delta(r+\delta)^3(w-\tau\tilde{x}_1)\,OKsofar
\end{aligned}
$$

4. The coefficient of $\tilde{x}_2$, β_1

$$
\begin{aligned}
=&+2\Psi_2 r(r+\delta)\left\{b\left[(r+\delta)^2+r(r+2\delta)\right]+rw(r+2\delta)-2\delta(r+\delta)(w-\tau\tilde{x}_1)+(r+\delta)\left[br-\delta(w-\tau\tilde{x}_1)\right]\right\}\\
&-2\Psi_2\left[2(r+\delta)^2+r^2\right]\left[rw(r+2\delta)+\delta^2(w-\tau\tilde{x}_1)\right]-r(r+\delta)(r+2\delta)\left[2\Psi_2 w(r+\delta)+\tau r(r+2\delta)\right]\\
=&2\Psi_2 br(r+\delta)\left[(r+\delta)^2+r(r+2\delta)\right]+2\Psi_2 r^2 w(r+\delta)(r+2\delta)-4\Psi_2 r\delta(r+\delta)^2(w-\tau\tilde{x}_1)+2\Psi_2 r(r+\delta)^2\left[br-\delta(w-\tau\tilde{x}_1)\right]\\
&-\left[4\Psi_2(r+\delta)^2+2\Psi_2 r^2\right]\left[rw(r+2\delta)+\delta^2(w-\tau\tilde{x}_1)\right]-2\Psi_2 wr(r+\delta)^2(r+2\delta)+\tau r^2(r+\delta)(r+2\delta)^2\\
=&2\Psi_2 br^2(r+\delta)(r+2\delta)+2\Psi_2 br(r+\delta)^3+2\Psi_2 r^2 w\delta(r+2\delta)+2\Psi_2 br^2(r+\delta)^2+\tau r^2(r+\delta)(r+2\delta)^2\\
&-6\Psi_2 r\delta(r+\delta)^2(w-\tau\tilde{x}_1)-6\Psi_2 rw(r+\delta)^2(r+2\delta)-4\Psi_2\delta^2(r+\delta)^2(w-\tau\tilde{x}_1)-2\Psi_2 r^2\delta^2(w-\tau\tilde{x}_1)
\end{aligned}
$$

5. The constant β_0

$$
=-r(r+\delta)(r+2\delta)^2\left[r^2(w-b)+r\delta(2w-b)+\delta(r+2\delta)(w-\tau\tilde{x}_1)\right]
$$